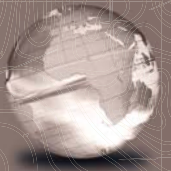


GLOBAL
EDITION



THOMAS' CALCULUS

Early Transcendentals

FIFTEENTH EDITION IN SI UNITS

Hass • Heil • Bogacki • Weir



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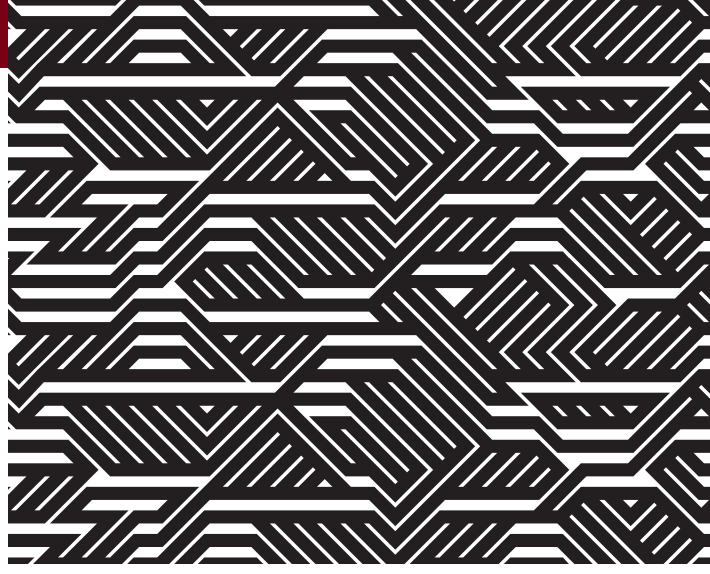
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1

Functions



OVERVIEW In this chapter we review what functions are and how they are visualized as graphs, how they are combined and transformed, and ways they can be classified.

1.1 Functions and Their Graphs

Functions are a tool for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description; we will use all four representations throughout this text. This section reviews these ideas.

Functions; Domain and Range

The temperature at which water boils depends on the elevation above sea level. The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels depends on the elapsed time.

In each case, the value of one variable quantity, say y , depends on the value of another variable quantity, which we often call x . We say that “ y is a function of x ” and write this symbolically as

$$y = f(x) \quad (\text{“}y \text{ equals } f \text{ of } x\text{”}).$$

The symbol f represents the function, the letter x is the **independent variable** representing the input value to f , and y is the **dependent variable** or output value of f at x .

DEFINITION A **function** f from a set D to a set Y is a rule that assigns a single value $f(x)$ in Y to each x in D .

A rule that assigns more than one value to an input x , such as the rule that assigns to a positive number both the positive and negative square roots of the number, does not describe a function.

The set D of all possible input values is called the **domain** of the function. The domain of f will sometimes be denoted by $D(f)$. The set of all output values $f(x)$ as x varies throughout D is called the **range** of the function. The range might not include every element in the set Y . The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line. (In Chapters 12–15, we will encounter functions for which the elements of the sets are points in the plane, or in space.)

Often a function is given by a formula that describes how to calculate the output value from the input variable. For instance, the equation $A = \pi r^2$ is a rule that calculates the area A of a circle from its radius r . When we define a function f with a formula $y = f(x)$ and the domain is not stated explicitly or restricted by context, the domain is assumed to be

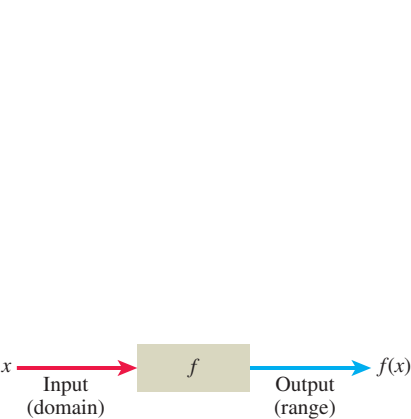


FIGURE 1.1 A diagram showing a function as a kind of machine.

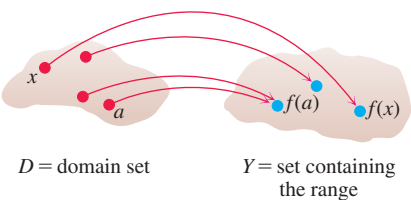


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D .

the largest set of real x -values for which the formula gives real y -values. This is called the **natural domain** of f . If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the domain of the function to, say, positive values of x , we would write “ $y = x^2, x > 0$.”

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2, x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In set notation (see Appendix A.1), the range is $\{x^2 \mid x \geq 2\}$ or $\{y \mid y \geq 4\}$ or $[4, \infty)$.

When the range of a function is a set of real numbers, the function is said to be **real-valued**. The domains and ranges of most real-valued functions we consider are intervals or combinations of intervals. Sometimes the range of a function is not easy to find.

A function f is like a machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Figure 1.1). The function keys on a calculator give an example of a function as a machine. For instance, whenever you enter a nonnegative number x and press the $\sqrt{x}$ key, the calculator gives an output value (the square root of x).

A function can also be pictured as an **arrow diagram** (Figure 1.2). Each arrow associates to an element of the domain D a single element in the set Y . In Figure 1.2, the arrows indicate that $f(a)$ is associated with a , $f(x)$ is associated with x , and so on. Notice that a function can have the same *output value* for two different input elements in the domain (as occurs with $f(a)$ in Figure 1.2), but each input element x is assigned a *single* output value $f(x)$.

EXAMPLE 1 Verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y -value for any real number x , so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root: $y = (\sqrt{y})^2$.

The formula $y = 1/x$ gives a real y -value for every x except $x = 0$. For consistency in the rules of arithmetic, we cannot divide any number by zero. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input that is assigned to the output value y .

The formula $y = \sqrt{x}$ gives a real y -value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number’s square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives nonnegative real y -values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y -value for every x in the closed interval from -1 to 1 . Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

Graphs of Functions

If f is a function with domain D , its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f . In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is the straight line sketched in Figure 1.3.

The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above (or below) the point x . The height may be positive or negative, depending on the sign of $f(x)$ (Figure 1.4).

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

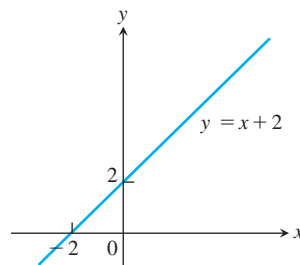


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

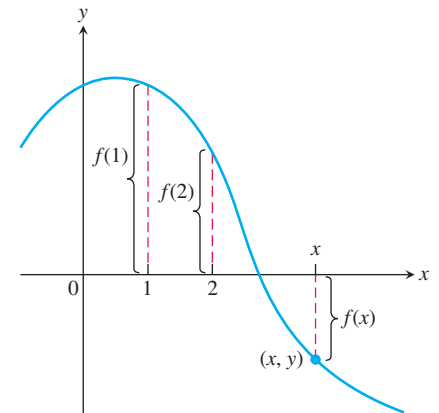


FIGURE 1.4 If (x, y) lies on the graph of f , then the value $y = f(x)$ is the height of the graph above (or below) the point x if $f(x)$ is negative).

EXAMPLE 2 Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution Make a table of xy -pairs that satisfy the equation $y = x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a *smooth* curve (labeled with its equation) through the plotted points (see Figure 1.5). ■

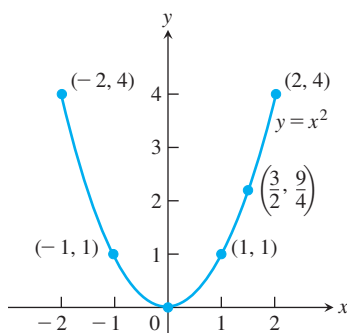
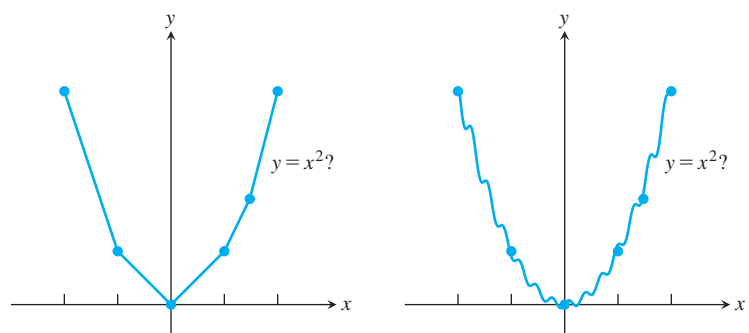


FIGURE 1.5 Graph of the function in Example 2.

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?



To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? Calculus answers this question, as we will see in Chapter 4. Meanwhile, we will have to settle for plotting points and connecting them as best we can.

Time	Pressure
0.00091	−0.080
0.00108	0.200
0.00125	0.480
0.00144	0.693
0.00162	0.816
0.00180	0.844
0.00198	0.771
0.00216	0.603
0.00234	0.368
0.00253	0.099
0.00271	−0.141
0.00289	−0.309
0.00307	−0.348
0.00325	−0.248
0.00344	−0.041
0.00362	0.217
0.00379	0.480
0.00398	0.681
0.00416	0.810
0.00435	0.827
0.00453	0.749
0.00471	0.581
0.00489	0.346
0.00507	0.077
0.00525	−0.164
0.00543	−0.320
0.00562	−0.354
0.00579	−0.248
0.00598	−0.035

Representing a Function Numerically

A function may be represented algebraically by a formula and visually by a graph (Example 2). Another way to represent a function is **numerically**, through a table of values. From an appropriate table of values, a graph of the function can be obtained using the method illustrated in Example 2, possibly with the aid of a computer. The graph consisting of only the points in the table is called a **scatterplot**.

EXAMPLE 3 Musical notes are pressure waves in the air. The data associated with Figure 1.6 give recorded pressure displacement versus time in seconds of a musical note produced by a tuning fork. The table provides a representation of the pressure function (in micropascals) over time. If we first make a scatterplot and then draw a smooth curve that approximates the data points (t, p) from the table, we obtain the graph shown in the figure.

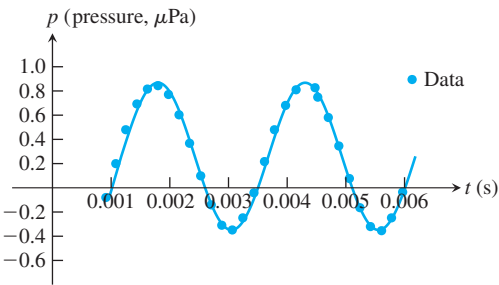


FIGURE 1.6 A smooth curve approximating the plotted points gives a graph of the pressure function represented by the accompanying tabled data (Example 3).

The Vertical Line Test for a Function

Not every curve in the coordinate plane can be the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so *no vertical line* can intersect the graph of a function at more than one point. If a is in the domain of the function f , then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

A circle cannot be the graph of a function, since some vertical lines intersect the circle twice. The circle graphed in Figure 1.7a, however, contains the graphs of two functions of x , namely the upper semicircle defined by the function $f(x) = \sqrt{1 - x^2}$ and the lower semicircle defined by the function $g(x) = -\sqrt{1 - x^2}$ (Figures 1.7b and 1.7c).

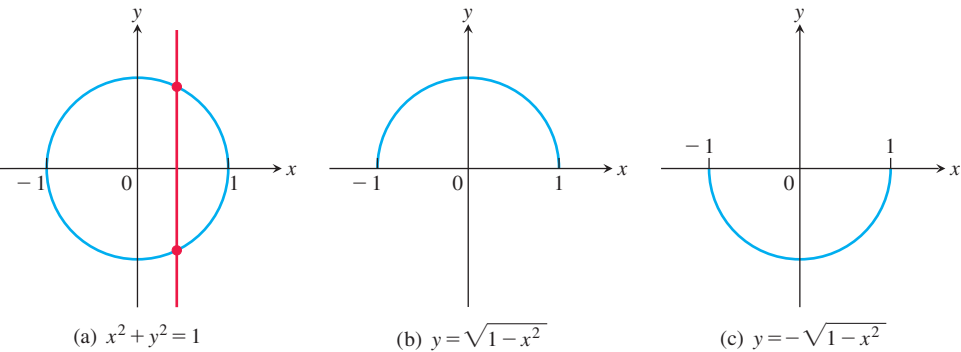


FIGURE 1.7 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of the function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of the function $g(x) = -\sqrt{1 - x^2}$.

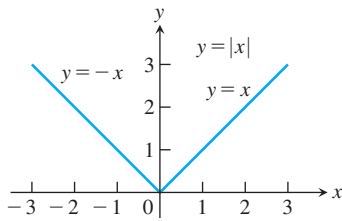


FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

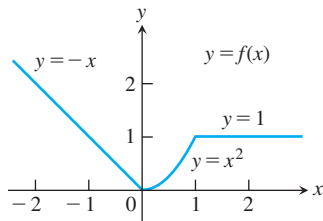


FIGURE 1.9 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 4).

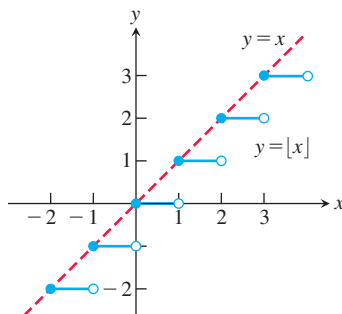


FIGURE 1.10 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 5).

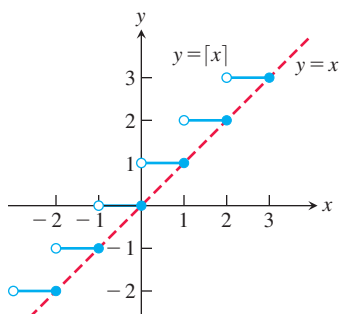


FIGURE 1.11 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x (Example 6).

Piecewise-Defined Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 & \text{First formula} \\ -x, & x < 0 & \text{Second formula} \end{cases}$$

whose graph is given in Figure 1.8. The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$. Piecewise-defined functions often arise when real-world data are modeled. Here are some other examples.

EXAMPLE 4 The function

$$f(x) = \begin{cases} -x, & x < 0 & \text{First formula} \\ x^2, & 0 \leq x \leq 1 & \text{Second formula} \\ 1, & x > 1 & \text{Third formula} \end{cases}$$

is defined on the entire real line but has values given by different formulas, depending on the position of x . The values of f are given by $y = -x$ when $x < 0$, $y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers (Figure 1.9).

EXAMPLE 5 The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$. Figure 1.10 shows the graph. Observe that

$$\begin{aligned} \lfloor 2.4 \rfloor &= 2, & \lfloor 1.9 \rfloor &= 1, & \lfloor 0 \rfloor &= 0, & \lfloor -1.2 \rfloor &= -2, \\ \lfloor 2 \rfloor &= 2, & \lfloor 0.2 \rfloor &= 0, & \lfloor -0.3 \rfloor &= -1, & \lfloor -2 \rfloor &= -2. \end{aligned}$$

EXAMPLE 6 The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 1.11 shows the graph. For positive values of x , this function might represent, for example, the cost of parking x hours in a parking lot that charges \$1 for each hour or part of an hour.

Increasing and Decreasing Functions

If the graph of a function climbs or rises as you move from left to right, we say that the function is *increasing*. If the graph descends or falls as you move from left to right, the function is *decreasing*.

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be two distinct points in I .

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I .
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I .

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for *every* pair of points x_1 and x_2 in I with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of $\leq$, it is sometimes said that f is *strictly increasing* or *strictly decreasing* on I . The interval I may be finite (also called bounded) or infinite (unbounded).

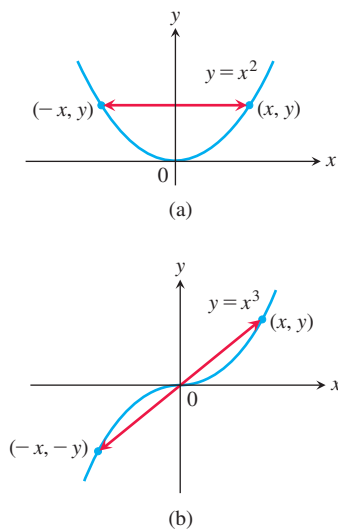


FIGURE 1.12 (a) The graph of $y = x^2$ (an even function) is symmetric about the y -axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

EXAMPLE 7 The function graphed in Figure 1.9 is decreasing on $(-\infty, 0)$ and increasing on $(0, 1)$. The function is neither increasing nor decreasing on the interval $(1, \infty)$ because the function is constant on that interval, and hence the strict inequalities in the definition of increasing or decreasing are not satisfied on $(1, \infty)$. ■

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have special symmetry properties.

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

The names *even* and *odd* come from powers of x . If y is an even power of x , as in $y = x^2$ or $y = x^4$, it is an even function of x because $(-x)^2 = x^2$ and $(-x)^4 = x^4$. If y is an odd power of x , as in $y = x$ or $y = x^3$, it is an odd function of x because $(-x)^1 = -x$ and $(-x)^3 = -x^3$.

The graph of an even function is **symmetric about the y -axis**. Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.12a). A reflection across the y -axis leaves the graph unchanged.

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.12b). Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged.

Notice that each of these definitions requires that both x and $-x$ be in the domain of f .

EXAMPLE 8 Here are several functions illustrating the definitions.

$f(x) = x^2$ Even function: $(-x)^2 = x^2$ for all x ; symmetry about y -axis. So $f(-3) = 9 = f(3)$. Changing the sign of x does not change the value of an even function.

$f(x) = x^2 + 1$ Even function: $(-x)^2 + 1 = x^2 + 1$ for all x ; symmetry about y -axis (Figure 1.13a).

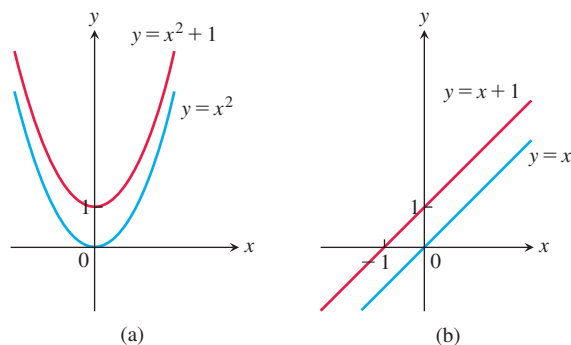


FIGURE 1.13 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y -axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd, since the symmetry about the origin is lost. The function $y = x + 1$ is also not even (Example 8).

$f(x) = x$	Odd function: $(-x) = -x$ for all x ; symmetry about the origin. So $f(-3) = -3$ while $f(3) = 3$. Changing the sign of x changes the sign of the value of an odd function.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal. Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.13b). ■

Common Functions

A variety of important types of functions are frequently encountered in calculus.

Linear Functions A function of the form $f(x) = mx + b$, where m and b are fixed constants, is called a **linear function**. Figure 1.14a shows an array of lines $f(x) = mx$. Each of these has $b = 0$, so these lines pass through the origin. The function $f(x) = x$, where $m = 1$ and $b = 0$, is called the **identity function**. Constant functions result when the slope is $m = 0$ (Figure 1.14b).

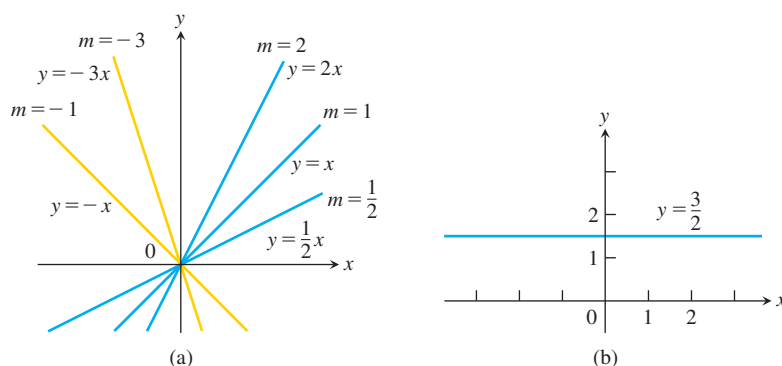


FIGURE 1.14 (a) Lines through the origin with slope m . (b) A constant function with slope $m = 0$.

DEFINITION Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other—that is, if $y = kx$ for some nonzero constant k .

If the variable y is proportional to the reciprocal $1/x$, then sometimes it is said that y is **inversely proportional** to x (because $1/x$ is the multiplicative inverse of x).

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $f(x) = x^a$ with $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 1.15. These functions are defined for all real values of x . Notice that as the power n gets larger, the curves tend to flatten toward the x -axis on the interval $(-1, 1)$ and to rise more steeply for $|x| > 1$. Each curve passes through the point $(1, 1)$ and through the origin. The graphs of functions with even powers are symmetric about the y -axis; those with odd powers are symmetric about the origin. The even-powered functions are decreasing on the interval $(-\infty, 0]$ and increasing on $[0, \infty)$; the odd-powered functions are increasing over the entire real line $(-\infty, \infty)$.

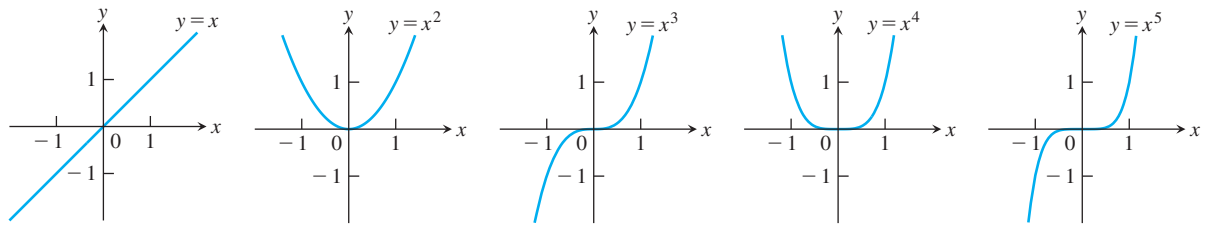


FIGURE 1.15 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

(b) $f(x) = x^a$ with $a = -1$ or $a = -2$.

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $f(x) = x^{-2} = 1/x^2$ are shown in Figure 1.16. Both functions are defined for all $x \neq 0$ (you can never divide by zero). The graph of $y = 1/x$ is the hyperbola $xy = 1$, which approaches the coordinate axes far from the origin. The graph of $y = 1/x^2$ also approaches the coordinate axes. The graph of the function $f(x) = 1/x$ is symmetric about the origin; this function is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. The graph of the function $f(x) = 1/x^2$ is symmetric about the y -axis; this function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

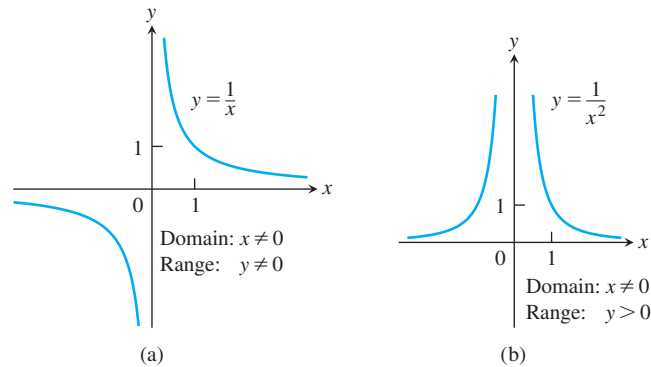


FIGURE 1.16 Graphs of the power functions $f(x) = x^a$.
(a) $a = -1$. (b) $a = -2$.

(c) $f(x) = x^a$ with $a = \frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{2}$, or $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $f(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x . Their graphs are displayed in Figure 1.17, along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

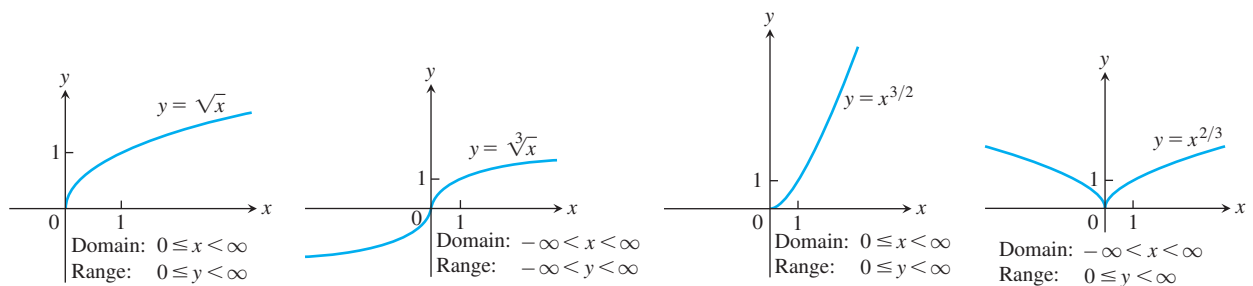


FIGURE 1.17 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{2}$, and $\frac{2}{3}$.

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \dots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then n is called the **degree** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions**. Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure 1.18 shows the graphs of three polynomials. Techniques to graph polynomials are studied in Chapter 4.

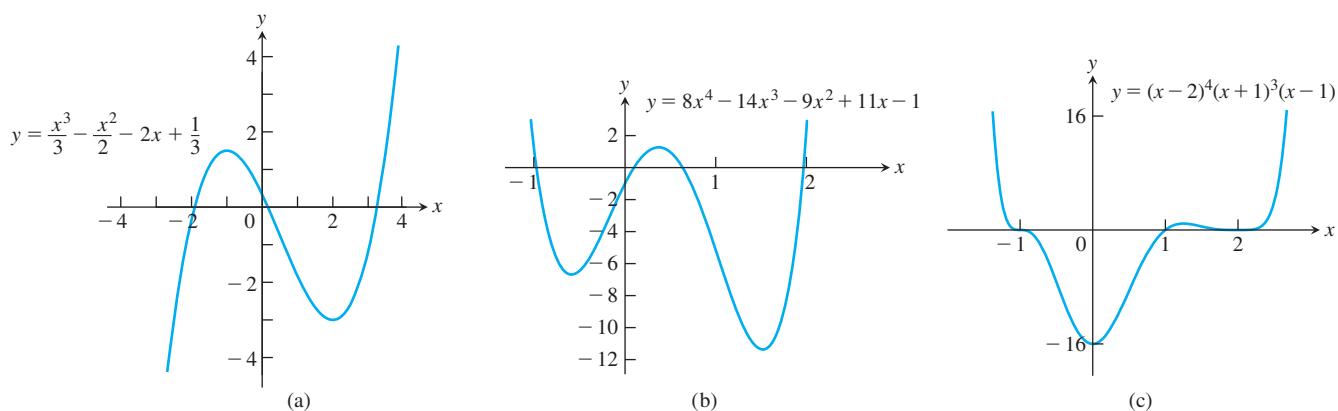


FIGURE 1.18 Graphs of three polynomial functions.

Rational Functions A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. The graphs of three rational functions are shown in Figure 1.19.

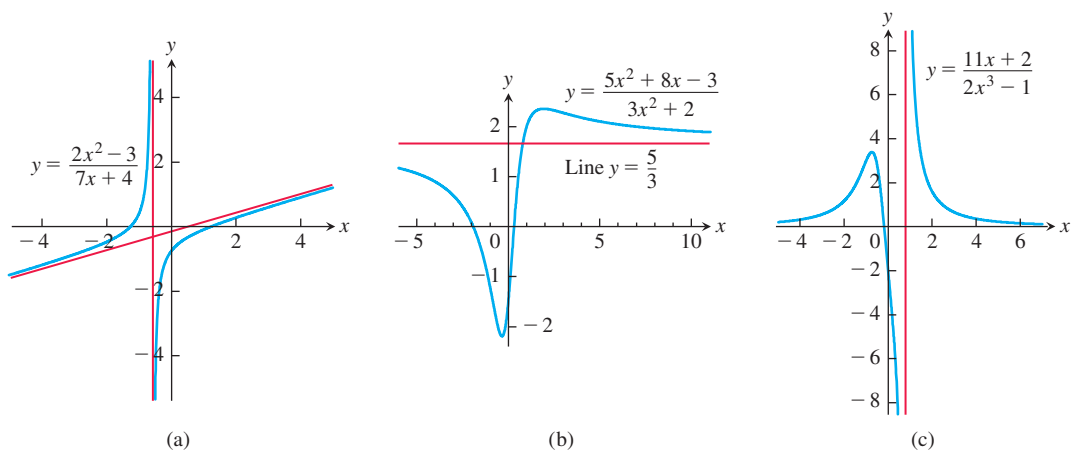


FIGURE 1.19 Graphs of three rational functions. The straight red lines approached by the graphs are called *asymptotes* and are not part of the graphs. We discuss asymptotes in Section 2.5.

Algebraic Functions Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of **algebraic functions**. All rational functions are algebraic, but also included are more

complicated functions (such as those satisfying an equation like $y^3 - 9xy + x^3 = 0$, studied in Section 3.7). Figure 1.20 displays the graphs of three algebraic functions.

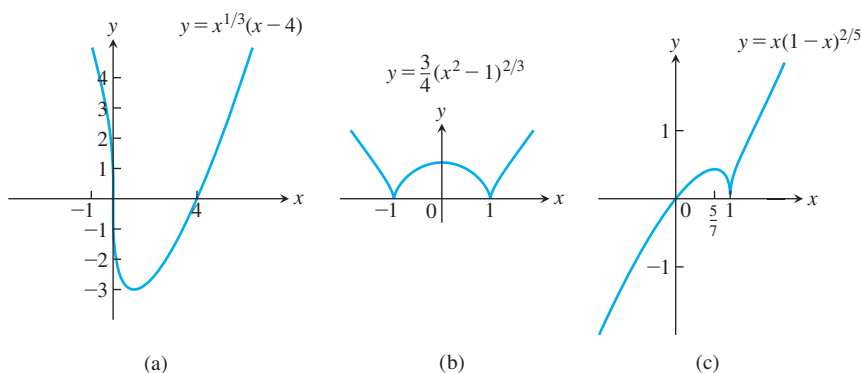


FIGURE 1.20 Graphs of three algebraic functions.

Trigonometric Functions The six basic trigonometric functions are reviewed in Section 1.3. The graphs of the sine and cosine functions are shown in Figure 1.21.

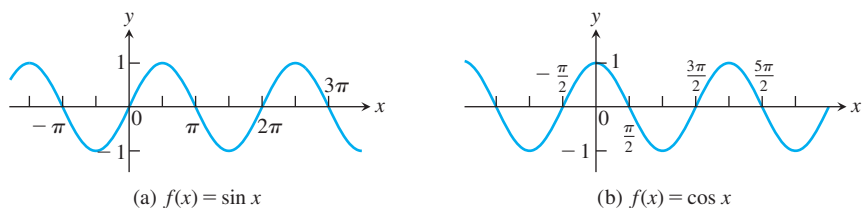


FIGURE 1.21 Graphs of the sine and cosine functions.

Exponential Functions A function of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is called an **exponential function** (with base a). All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0. We discuss exponential functions in Section 1.4. The graphs of some exponential functions are shown in Figure 1.22.

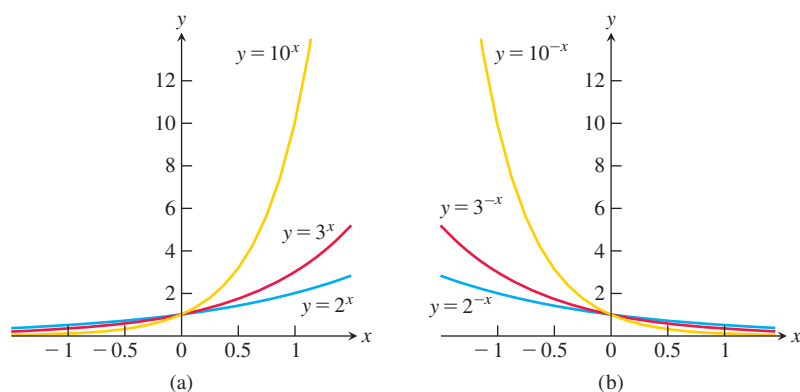


FIGURE 1.22 Graphs of exponential functions.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions, and we discuss these functions in Section 1.5. Figure 1.23 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

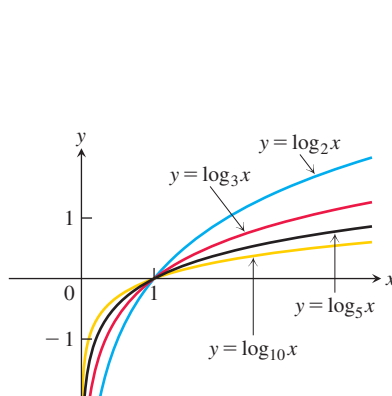


FIGURE 1.23 Graphs of four logarithmic functions.

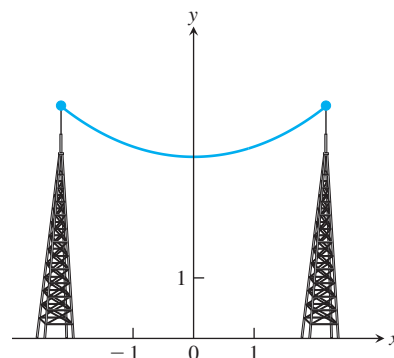


FIGURE 1.24 Graph of a catenary or hanging cable. (The Latin word *catena* means “chain.”)

Transcendental Functions These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well. The **catenary** is one example of a transcendental function. Its graph has the shape of a cable, like a telephone line or electric cable, strung from one support to another and hanging freely under its own weight (Figure 1.24). The function defining the graph is discussed in Section 7.3.

EXERCISES 1.1

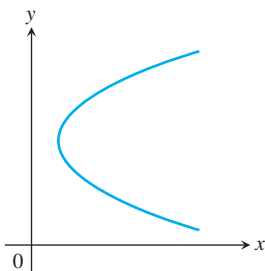
Functions

In Exercises 1–6, find the domain and range of each function.

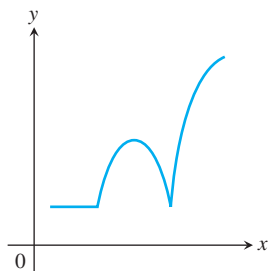
1. $f(x) = 1 + x^2$
2. $f(x) = 1 - \sqrt{x}$
3. $F(x) = \sqrt{5x + 10}$
4. $g(x) = \sqrt{x^2 - 3x}$
5. $f(t) = \frac{4}{3 - t}$
6. $G(t) = \frac{2}{t^2 - 16}$

In Exercises 7 and 8, which of the graphs are graphs of functions of x , and which are not? Give reasons for your answers.

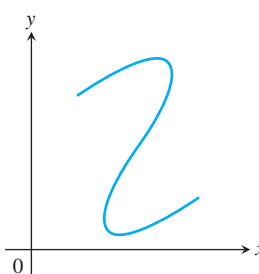
7. a.



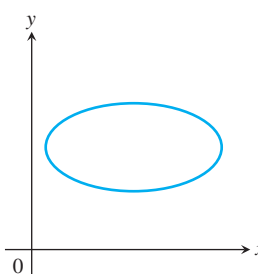
b.



8. a.



b.



Finding Formulas for Functions

9. Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x .
10. Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.
11. Express the edge length of a cube as a function of the cube's diagonal length d . Then express the surface area and volume of the cube as a function of the diagonal length.

12. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt{x}$. Express the coordinates of P as functions of the slope of the line joining P to the origin.
13. Consider the point (x, y) lying on the graph of the line $2x + 4y = 5$. Let L be the distance from the point (x, y) to the origin $(0, 0)$. Write L as a function of x .
14. Consider the point (x, y) lying on the graph of $y = \sqrt{x - 3}$. Let L be the distance between the points (x, y) and $(4, 0)$. Write L as a function of y .

Functions and Graphs

Find the natural domain and graph the functions in Exercises 15–20.

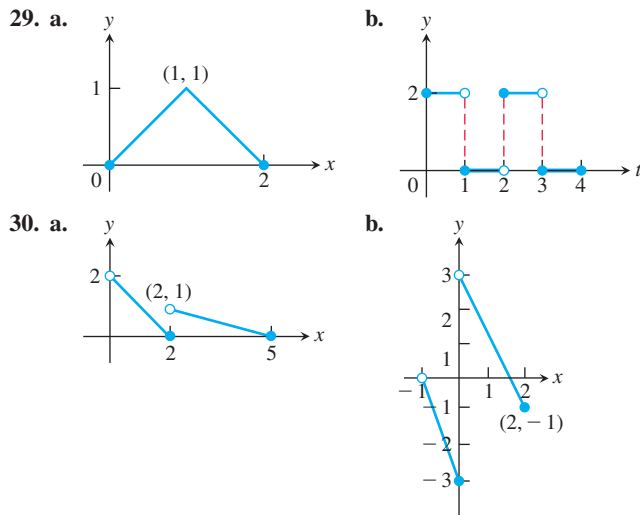
15. $f(x) = 5 - 2x$ 16. $f(x) = 1 - 2x - x^2$
 17. $g(x) = \sqrt{|x|}$ 18. $g(x) = \sqrt{-x}$
 19. $F(t) = t/|t|$ 20. $G(t) = 1/|t|$
21. Find the domain of $y = \frac{x+3}{4-\sqrt{x^2-9}}$.
22. Find the range of $y = 2 + \sqrt{9+x^2}$.
23. Graph the following equations and explain why they are not graphs of functions of x .
 a. $|y| = x$ b. $y^2 = x^2$
24. Graph the following equations and explain why they are not graphs of functions of x .
 a. $|x| + |y| = 1$ b. $|x + y| = 1$

Piecewise-Defined Functions

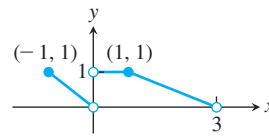
Graph the functions in Exercises 25–28.

25. $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$
 26. $g(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$
 27. $F(x) = \begin{cases} 4 - x^2, & x \leq 1 \\ x^2 + 2x, & x > 1 \end{cases}$
 28. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \leq x \end{cases}$

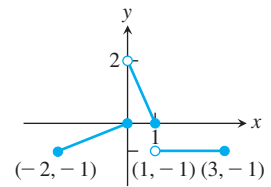
Find a formula for each function graphed in Exercises 29–32.



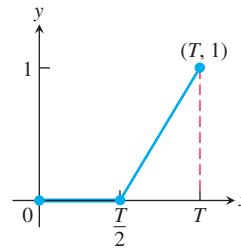
31. a.



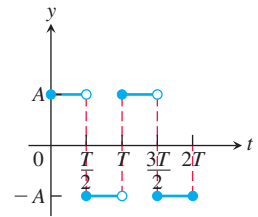
b.



32. a.



b.



The Greatest and Least Integer Functions

33. For what values of x is
 a. $\lfloor x \rfloor = 0$? b. $\lceil x \rceil = 0$?
34. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?
35. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x ? Give reasons for your answer.
36. Graph the function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x \geq 0 \\ \lceil x \rceil, & x < 0. \end{cases}$$

Why is $f(x)$ called the *integer part* of x ?

Increasing and Decreasing Functions

Graph the functions in Exercises 37–46. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

37. $y = -x^3$ 38. $y = -\frac{1}{x^2}$
 39. $y = -\frac{1}{x}$ 40. $y = \frac{1}{|x|}$
 41. $y = \sqrt{|x|}$ 42. $y = \sqrt{-x}$
 43. $y = x^3/8$ 44. $y = -4\sqrt{x}$
 45. $y = -x^{3/2}$ 46. $y = (-x)^{2/3}$

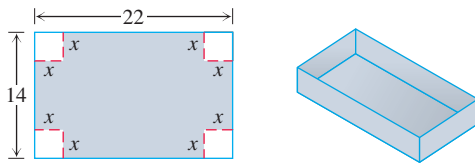
Even and Odd Functions

In Exercises 47–62, say whether the function is even, odd, or neither. Give reasons for your answer.

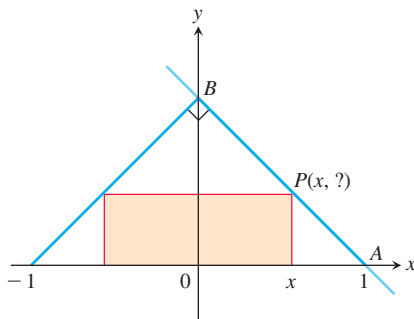
47. $f(x) = 3$ 48. $f(x) = x^{-5}$
 49. $f(x) = x^2 + 1$ 50. $f(x) = x^2 + x$
 51. $g(x) = x^3 + x$ 52. $g(x) = x^4 + 3x^2 - 1$
 53. $g(x) = \frac{1}{x^2 - 1}$ 54. $g(x) = \frac{x}{x^2 - 1}$
 55. $h(t) = \frac{1}{t - 1}$ 56. $h(t) = |t^3|$
 57. $h(t) = 2t + 1$ 58. $h(t) = 2|t| + 1$
 59. $\sin 2x$ 60. $\sin x^2$
 61. $\cos 3x$ 62. $1 + \cos x$

Theory and Examples

63. The variable s is proportional to t , and $s = 25$ when $t = 75$. Determine t when $s = 60$.
64. **Kinetic energy** The kinetic energy K of a mass is proportional to the square of its velocity v . If $K = 12,960$ joules when $v = 18$ m/s, what is K when $v = 10$ m/s?
65. The variables r and s are inversely proportional, and $r = 6$ when $s = 4$. Determine s when $r = 10$.
66. **Boyle's law** Boyle's law says that the volume V of a gas at constant temperature increases whenever the pressure P decreases, so that V and P are inversely proportional. If $P = 14.7$ N/cm² when $V = 1000$ cm³, then what is V when $P = 23.4$ N/cm²?
67. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 14 cm by 22 cm by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x .

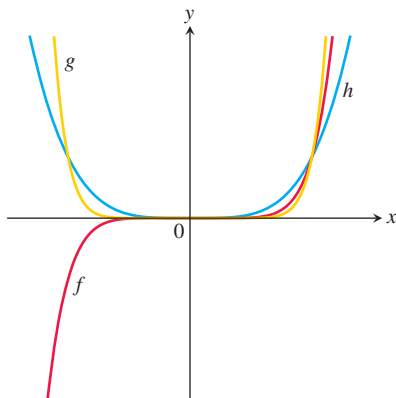


68. The accompanying figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.
- Express the y -coordinate of P in terms of x . (You might start by writing an equation for the line AB .)
 - Express the area of the rectangle in terms of x .



In Exercises 69 and 70, match each equation with its graph. Do not use a graphing device, and give reasons for your answer.

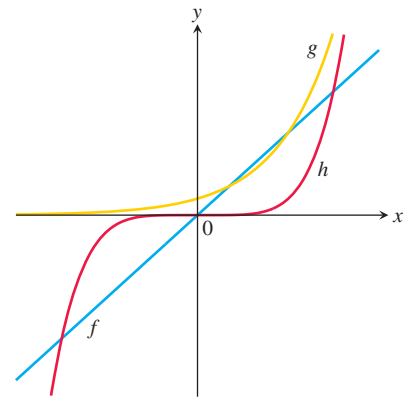
69. a. $y = x^4$ b. $y = x^7$ c. $y = x^{10}$



70. a. $y = 5x$

b. $y = 5^x$

c. $y = x^5$



71. a. Graph the functions $f(x) = x/2$ and $g(x) = 1 + (4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1 + \frac{4}{x}.$$

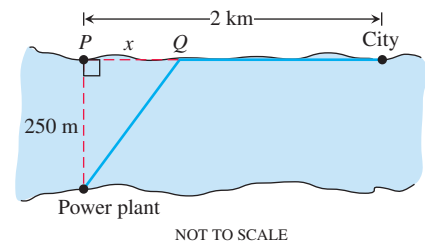
- b. Confirm your findings in part (a) algebraically.

72. a. Graph the functions $f(x) = 3/(x - 1)$ and $g(x) = 2/(x + 1)$ together to identify the values of x for which

$$\frac{3}{x - 1} < \frac{2}{x + 1}.$$

- b. Confirm your findings in part (a) algebraically.

73. For a curve to be *symmetric about the x -axis*, the point (x, y) must lie on the curve if and only if the point $(x, -y)$ lies on the curve. Explain why a curve that is symmetric about the x -axis is not the graph of a function, unless the function is $y = 0$.
74. Three hundred books sell for \$40 each, resulting in a revenue of $(300)(\$40) = \$12,000$. For each \$5 increase in the price, 25 fewer books are sold. Write the revenue R as a function of the number x of \$5 increases.
75. A pen in the shape of an isosceles right triangle with legs of length x m and hypotenuse of length h m is to be built. If fencing costs \$5/m for the legs and \$10/m for the hypotenuse, write the total cost C of construction as a function of h .
76. **Industrial costs** A power plant sits next to a river where the river is 250 m wide. To lay a new cable from the plant to a location in the city 2 km downstream on the opposite side costs \$180 per meter across the river and \$100 per meter along the land.



- Suppose that the cable goes from the plant to a point Q on the opposite side that is x m from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x .
- Generate a table of values to determine if the least expensive location for point Q is less than 300 m or greater than 300 m from point P .

1.2 Combining Functions; Shifting and Scaling Graphs

In this section we look at the main ways functions are combined or transformed to form new functions.

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, $f - g$, and fg by the formulas

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ (f - g)(x) &= f(x) - g(x) \\ (fg)(x) &= f(x)g(x).\end{aligned}$$

Notice that the $+$ sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the $+$ on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 1 The functions defined by the formulas

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{1 - x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points in

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg .

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)(x = 1 \text{ excluded})$
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1](x = 0 \text{ excluded})$

The graph of the function $f + g$ is obtained from the graphs of f and g by adding the corresponding y -coordinates $f(x)$ and $g(x)$ at each point $x \in D(f) \cap D(g)$, as in Figure 1.25. The graphs of $f + g$ and $f \cdot g$ from Example 1 are shown in Figure 1.26.

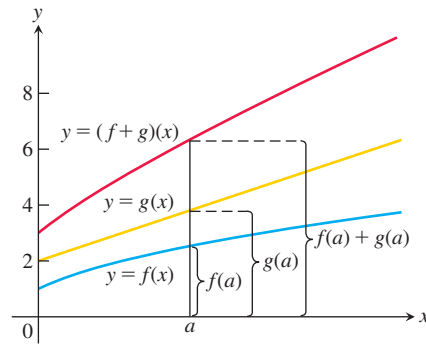


FIGURE 1.25 Graphical addition of two functions.

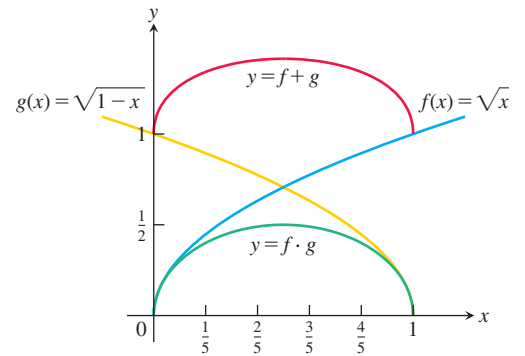


FIGURE 1.26 The domain of the function $f + g$ is the intersection of the domains of f and g , the interval $[0, 1]$ on the x -axis where these domains overlap. This interval is also the domain of the function $f \cdot g$ (Example 1).

Composing Functions

Composition is another method for combining functions. In this operation the output from one function becomes the input to a second function.

DEFINITION If f and g are functions, the function $f \circ g$ (“ f composed with g ”) is defined by

$$(f \circ g)(x) = f(g(x))$$

and called the **composition** of f and g . The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f .

To find $(f \circ g)(x)$, *first* find $g(x)$ and *second* find $f(g(x))$. Figure 1.27 pictures $f \circ g$ as a machine diagram, and Figure 1.28 shows the composition as an arrow diagram.

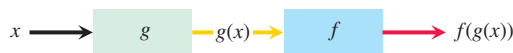


FIGURE 1.27 The composition $f \circ g$ uses the output $g(x)$ of the first function g as the input for the second function f .

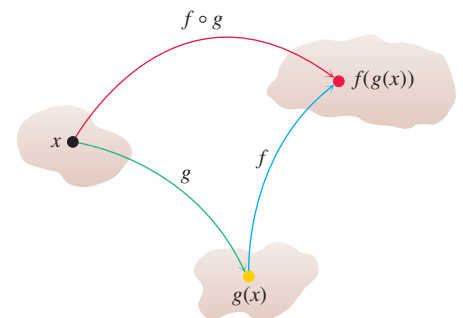


FIGURE 1.28 Arrow diagram for $f \circ g$. If x lies in the domain of g and $g(x)$ lies in the domain of f , then the functions f and g can be composed to form $(f \circ g)(x)$.

To evaluate the composition $g \circ f$ (when defined), we find $f(x)$ first and then find $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g .

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 2 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

- (a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$ (c) $(f \circ f)(x)$ (d) $(g \circ g)(x)$.

Solution

Composition	Domain
(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x+2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but $g(x)$ belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = (\sqrt{x})^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$, since $\sqrt{x}$ requires $x \geq 0$.

Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

Shift Formulas

Vertical Shifts

$y = f(x) + k$ Shifts the graph of f up k units if $k > 0$
Shifts it down $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ Shifts the graph of f left h units if $h > 0$
Shifts it right $|h|$ units if $h < 0$

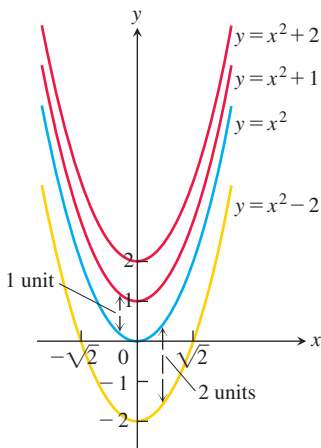


FIGURE 1.29 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Examples 3a and b).

EXAMPLE 3

- (a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.29).
 (b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.29).
 (c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left, while adding -2 shifts the graph 2 units to the right (Figure 1.30).
 (d) Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure 1.31). ■

Scaling and Reflecting a Graph of a Function

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function f , or the independent variable x , by an appropriate constant c . Reflections across the coordinate axes are special cases where $c = -1$.

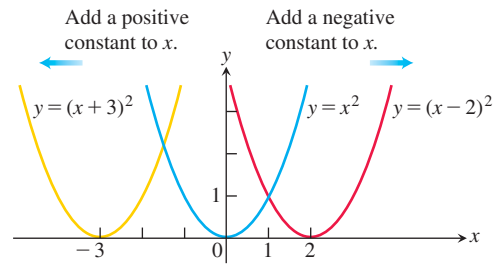


FIGURE 1.30 To shift the graph of $y = x^2$ to the left, we add a positive constant to x (Example 3c). To shift the graph to the right, we add a negative constant to x .

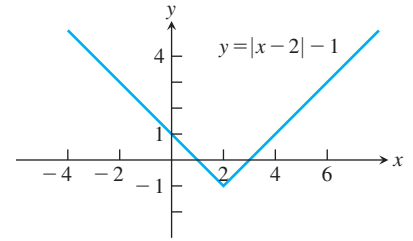


FIGURE 1.31 The graph of $y = |x|$ shifted 2 units to the right and 1 unit down (Example 3d).

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c .
$y = \frac{1}{c}f(x)$	Compresses the graph of f vertically by a factor of c .
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c .
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c .

For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x -axis.
$y = f(-x)$	Reflects the graph of f across the y -axis.

EXAMPLE 4 Here we scale and reflect the graph of $y = \sqrt{x} + 1$.

- (a) **Vertical:** Multiplying the right-hand side of $y = \sqrt{x} + 1$ by 3 to get $y = 3(\sqrt{x} + 1)$ stretches the graph vertically by a factor of 3, whereas multiplying by $1/3$ compresses the graph vertically by a factor of 3 (Figure 1.32).
- (b) **Horizontal:** The graph of $y = \sqrt{3x} + 1$ is a horizontal compression of the graph of $y = \sqrt{x} + 1$ by a factor of 3, and $y = \sqrt{x/3} + 1$ is a horizontal stretching by a factor of 3 (Figure 1.33).
- (c) **Reflection:** The graph of $y = -(\sqrt{x} + 1)$ is a reflection of $y = \sqrt{x} + 1$ across the x -axis, and $y = \sqrt{-x} + 1$ is a reflection across the y -axis (Figure 1.34). ■

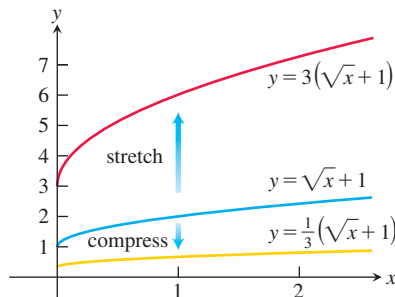


FIGURE 1.32 Vertically stretching and compressing the graph of $y = \sqrt{x} + 1$ by a factor of 3 (Example 4a).

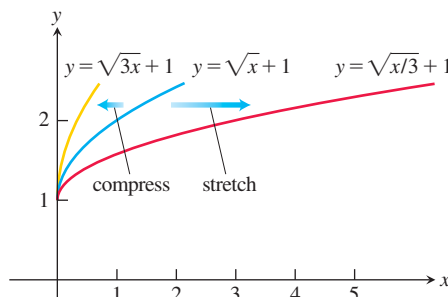


FIGURE 1.33 Horizontally stretching and compressing the graph of $y = \sqrt{x} + 1$ by a factor of 3 (Example 4b).

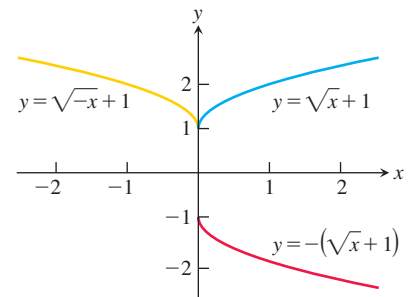


FIGURE 1.34 Reflections of the graph of $y = \sqrt{x} + 1$ across the coordinate axes (Example 4c).

EXAMPLE 5 Given the function $f(x) = x^4 - 4x^3 + 10$ (Figure 1.35a), find formulas for the graphs resulting from

- (a) horizontal compression by a factor of 2 followed by reflection across the y -axis (Figure 1.35b).
- (b) vertical compression by a factor of 2 followed by reflection across the x -axis (Figure 1.35c).

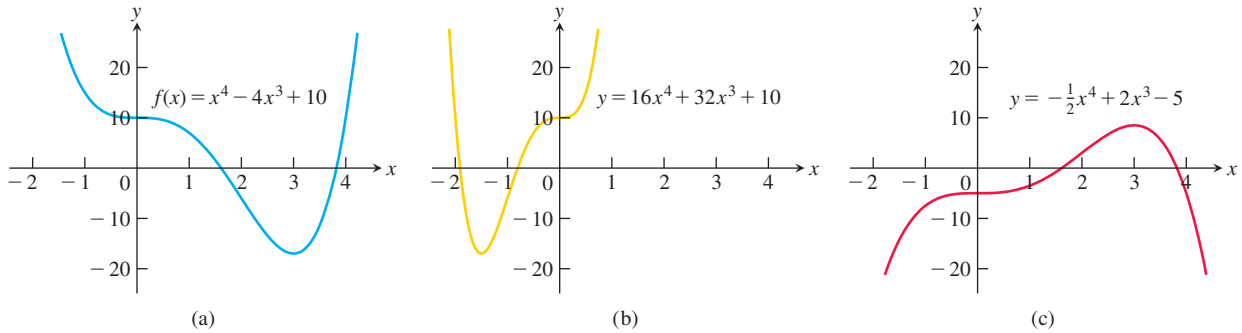


FIGURE 1.35 (a) The original graph of f . (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the y -axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the x -axis (Example 5).

Solution

- (a) We multiply x by 2 to get the horizontal compression, and by -1 to give reflection across the y -axis. The formula is obtained by substituting $-2x$ for x in the right-hand side of the equation for f :

$$\begin{aligned} y &= f(-2x) = (-2x)^4 - 4(-2x)^3 + 10 \\ &= 16x^4 + 32x^3 + 10. \end{aligned}$$

- (b) The formula is

$$y = -\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5. \quad \blacksquare$$

EXERCISES 1.2

Algebraic Combinations

In Exercises 1 and 2, find the domains of f , g , $f + g$, and $f \cdot g$.

1. $f(x) = x$, $g(x) = \sqrt{x-1}$
2. $f(x) = \sqrt{x+1}$, $g(x) = \sqrt{x-1}$

In Exercises 3 and 4, find the domains of f , g , f/g , and g/f .

3. $f(x) = 2$, $g(x) = x^2 + 1$
4. $f(x) = 1$, $g(x) = 1 + \sqrt{x}$

Compositions of Functions

5. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

- | | |
|---------------|--------------|
| a. $f(g(0))$ | b. $g(f(0))$ |
| c. $f(g(x))$ | d. $g(f(x))$ |
| e. $f(f(-5))$ | f. $g(g(2))$ |
| g. $f(f(x))$ | h. $g(g(x))$ |

6. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.

- | | |
|----------------|----------------|
| a. $f(g(1/2))$ | b. $g(f(1/2))$ |
| c. $f(g(x))$ | d. $g(f(x))$ |
| e. $f(f(2))$ | f. $g(g(2))$ |
| g. $f(f(x))$ | h. $g(g(x))$ |

In Exercises 7–10, write a formula for $f \circ g \circ h$.

7. $f(x) = x + 1$, $g(x) = 3x$, $h(x) = 4 - x$

8. $f(x) = 3x + 4$, $g(x) = 2x - 1$, $h(x) = x^2$

9. $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x+4}$, $h(x) = \frac{1}{x}$

10. $f(x) = \frac{x+2}{3-x}$, $g(x) = \frac{x^2}{x^2+1}$, $h(x) = \sqrt{2-x}$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 11 and 12 as a composition involving one or more of f , g , h , and j .

11. a. $y = \sqrt{x} - 3$

b. $y = 2\sqrt{x}$

c. $y = x^{1/4}$

d. $y = 4x$

e. $y = \sqrt{(x-3)^3}$

f. $y = (2x-6)^3$

12. a. $y = 2x - 3$

b. $y = x^{3/2}$

c. $y = x^9$

d. $y = x - 6$

e. $y = 2\sqrt{x-3}$

f. $y = \sqrt{x^3 - 3}$

13. Copy and complete the following table.

$g(x)$	$f(x)$	$(f \circ g)(x)$
a. $x - 7$	$\sqrt{x}$	?
b. $x + 2$	$3x$	?
c. ?	$\sqrt{x-5}$	$\sqrt{x^2-5}$
d. $\frac{x}{x-1}$	$\frac{x}{x-1}$	?
e. ?	$1 + \frac{1}{x}$	x
f. $\frac{1}{x}$	?	x

14. Copy and complete the following table.

$g(x)$	$f(x)$	$(f \circ g)(x)$
a. $\frac{1}{x-1}$	$ x $	?
b. ?	$\frac{x-1}{x}$	$\frac{x}{x+1}$
c. ?	$\sqrt{x}$	$ x $
d. $\sqrt{x}$	?	$ x $

15. Evaluate each expression using the given table of values:

x	-2	-1	0	1	2
$f(x)$	1	0	-2	1	2
$g(x)$	2	1	0	-1	0

a. $f(g(-1))$

b. $g(f(0))$

c. $f(f(-1))$

d. $g(g(2))$

e. $g(f(-2))$

f. $f(g(1))$

16. Evaluate each expression using the functions

$$f(x) = 2 - x, \quad g(x) = \begin{cases} -x, & -2 \leq x < 0 \\ x - 1, & 0 \leq x \leq 2. \end{cases}$$

a. $f(g(0))$

b. $g(f(3))$

c. $g(g(-1))$

d. $f(f(2))$

e. $g(f(0))$

f. $f(g(1/2))$

In Exercises 17 and 18, (a) write formulas for $f \circ g$ and $g \circ f$ and (b) find the domain of each.

17. $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x}$

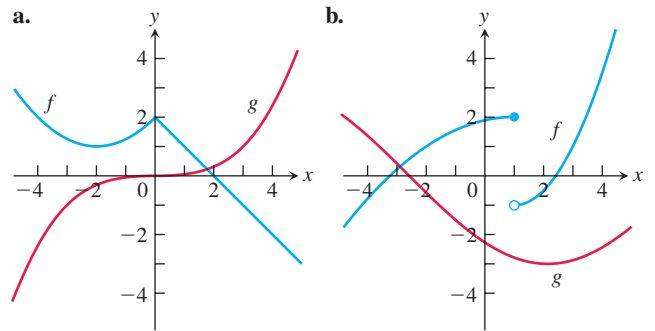
18. $f(x) = x^2$, $g(x) = 1 - \sqrt{x}$

19. Let $f(x) = \frac{x}{x-2}$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x$.

20. Let $f(x) = 2x^3 - 4$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x + 2$.

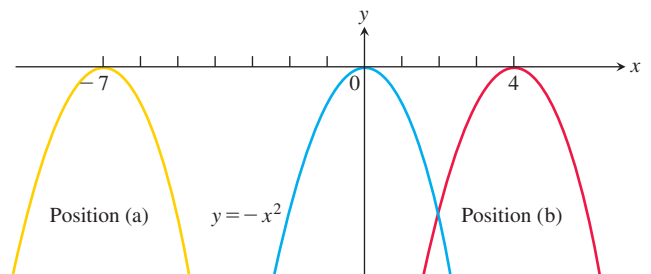
21. A balloon's volume V is given by $V = s^2 + 2s + 3 \text{ cm}^3$, where s is the ambient temperature in $^{\circ}\text{C}$. The ambient temperature s at time t minutes is given by $s = 2t - 3^{\circ}\text{C}$. Write the balloon's volume V as a function of time t .

22. Use the graphs of f and g to sketch the graph of $y = f(g(x))$.

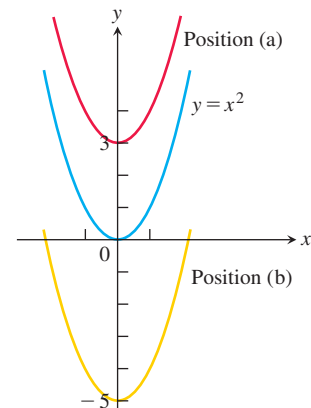


Shifting Graphs

23. The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.

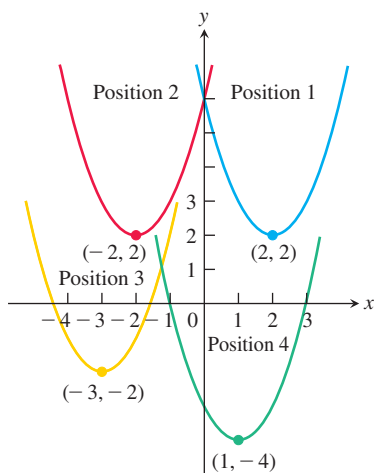


24. The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.

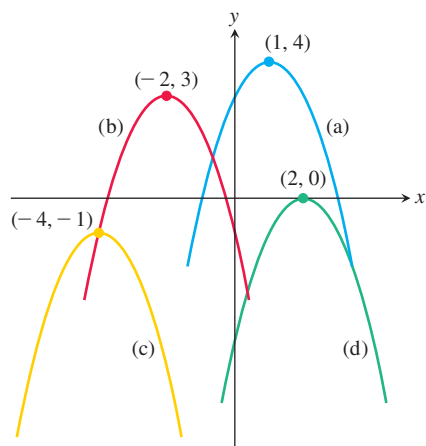


25. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

- a. $y = (x - 1)^2 - 4$ b. $y = (x - 2)^2 + 2$
 c. $y = (x + 2)^2 + 2$ d. $y = (x + 3)^2 - 2$



26. The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



Exercises 27–36 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

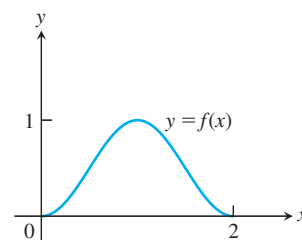
27. $x^2 + y^2 = 49$ Down 3, left 2
 28. $x^2 + y^2 = 25$ Up 3, left 4
 29. $y = x^3$ Left 1, down 1
 30. $y = x^{2/3}$ Right 1, down 1
 31. $y = \sqrt{x}$ Left 0.81
 32. $y = -\sqrt{x}$ Right 3
 33. $y = 2x - 7$ Up 7
 34. $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1
 35. $y = 1/x$ Up 1, right 1
 36. $y = 1/x^2$ Left 2, down 1

Graph the functions in Exercises 37–56.

37. $y = \sqrt{x + 4}$ 38. $y = \sqrt{9 - x}$

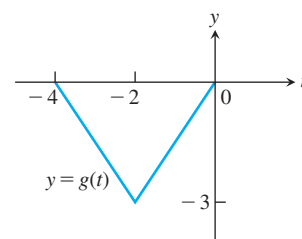
39. $y = |x - 2|$ 40. $y = |1 - x| - 1$
 41. $y = 1 + \sqrt{x - 1}$ 42. $y = 1 - \sqrt{x}$
 43. $y = (x + 1)^{2/3}$ 44. $y = (x - 8)^{2/3}$
 45. $y = 1 - x^{2/3}$ 46. $y + 4 = x^{2/3}$
 47. $y = \sqrt[3]{x - 1} - 1$ 48. $y = (x + 2)^{3/2} + 1$
 49. $y = \frac{1}{x - 2}$ 50. $y = \frac{1}{x} - 2$
 51. $y = \frac{1}{x} + 2$ 52. $y = \frac{1}{x + 2}$
 53. $y = \frac{1}{(x - 1)^2}$ 54. $y = \frac{1}{x^2} - 1$
 55. $y = \frac{1}{x^2} + 1$ 56. $y = \frac{1}{(x + 1)^2}$

57. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



- a. $f(x) + 2$ b. $f(x) - 1$
 c. $2f(x)$ d. $-f(x)$
 e. $f(x + 2)$ f. $f(x - 1)$
 g. $f(-x)$ h. $-f(x + 1) + 1$

58. The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



- a. $g(-t)$ b. $-g(t)$
 c. $g(t) + 3$ d. $1 - g(t)$
 e. $g(-t + 2)$ f. $g(t - 2)$
 g. $g(1 - t)$ h. $-g(t - 4)$

Vertical and Horizontal Scaling

Exercises 59–68 tell in what direction and by what factor the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

59. $y = x^2 - 1$, stretched vertically by a factor of 3
 60. $y = x^2 - 1$, compressed horizontally by a factor of 2
 61. $y = 1 + \frac{1}{x^2}$, compressed vertically by a factor of 2

62. $y = 1 + \frac{1}{x^2}$, stretched horizontally by a factor of 3
63. $y = \sqrt{x+1}$, compressed horizontally by a factor of 4
64. $y = \sqrt{x+1}$, stretched vertically by a factor of 3
65. $y = \sqrt{4-x^2}$, stretched horizontally by a factor of 2
66. $y = \sqrt{4-x^2}$, compressed vertically by a factor of 3
67. $y = 1 - x^3$, compressed horizontally by a factor of 3
68. $y = 1 - x^3$, stretched horizontally by a factor of 2

Graphing

In Exercises 69–76, graph each function not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.14–1.17 and applying an appropriate transformation.

69. $y = -\sqrt{2x+1}$ 70. $y = \sqrt{1-\frac{x}{2}}$
71. $y = (x-1)^3 + 2$ 72. $y = (1-x)^3 + 2$
73. $y = \frac{1}{2x} - 1$ 74. $y = \frac{2}{x^2} + 1$

75. $y = -\sqrt[3]{x}$ 76. $y = (-2x)^{2/3}$
77. Graph the function $y = |x^2 - 1|$.
78. Graph the function $y = \sqrt{|x|}$.

Combining Functions

79. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line $(-\infty, \infty)$. Which of the following (where defined) are even? odd?
- a. fg b. f/g c. g/f
- d. $f^2 = ff$ e. $g^2 = gg$ f. $f \circ g$
- g. $g \circ f$ h. $f \circ f$ i. $g \circ g$
80. Can a function be both even and odd? Give reasons for your answer.
- T** 81. (Continuation of Example 1.) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.
- T** 82. Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

1.3 Trigonometric Functions

This section reviews radian measure and the basic trigonometric functions.

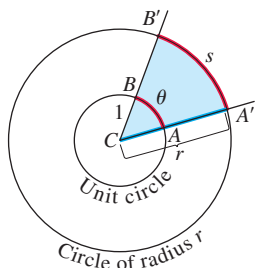


FIGURE 1.36 The radian measure of the central angle $A'CB'$ is the number $\theta = s/r$. For a unit circle of radius $r = 1$, θ is the length of arc AB that central angle ACB cuts from the unit circle.

Angles

Angles are measured in degrees or radians. The number of **radians** in the central angle $A'CB'$ within a circle of radius r is defined as the number of “radius units” contained in the arc s subtended by that central angle. If we denote this central angle by θ when measured in radians, this means that $\theta = s/r$ (Figure 1.36), or

$$s = r\theta \quad (\theta \text{ in radians}). \quad (1)$$

If the circle is a unit circle having radius $r = 1$, then from Figure 1.36 and Equation (1), we see that the central angle θ measured in radians is just the length of the arc that the angle cuts from the unit circle. Since one complete revolution of the unit circle is 360° or 2π radians, we have

$$\pi \text{ radians} = 180^\circ \quad (2)$$

and

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \quad \text{or} \quad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians}.$$

Table 1.1 shows the equivalence between degree and radian measures for some basic angles.

TABLE 1.1 Angles measured in degrees and radians

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

An angle in the xy -plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x -axis (Figure 1.37). Angles measured counterclockwise from the positive x -axis are assigned positive measures; angles measured clockwise are assigned negative measures.

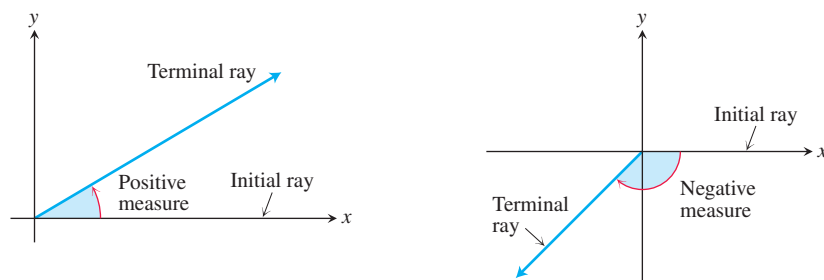


FIGURE 1.37 Angles in standard position in the xy -plane.

Angles describing counterclockwise rotations can go arbitrarily far beyond 2π radians or 360° . Similarly, angles describing clockwise rotations can have negative measures of all sizes (Figure 1.38).

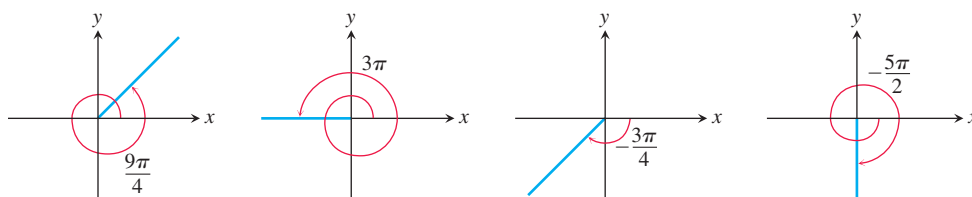


FIGURE 1.38 Nonzero radian measures can be positive or negative and can go beyond 2π .

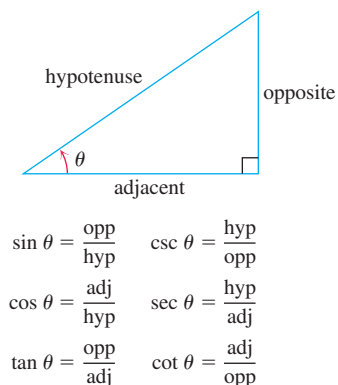


FIGURE 1.39 Trigonometric ratios of an acute angle.

Angle Convention: Use Radians From now on in this text, it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. Using radians simplifies many of the operations and computations in calculus.

The Six Basic Trigonometric Functions

The trigonometric functions of an acute angle are given in terms of the sides of a right triangle (Figure 1.39). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r . We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Figure 1.40).

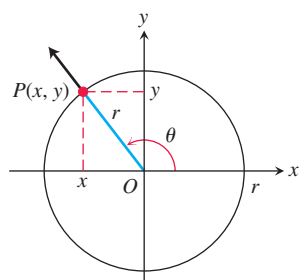


FIGURE 1.40 The trigonometric functions of a general angle θ are defined in terms of x , y , and r .

sine:	$\sin \theta = \frac{y}{r}$	cosecant:	$\csc \theta = \frac{r}{y}$
cosine:	$\cos \theta = \frac{x}{r}$	secant:	$\sec \theta = \frac{r}{x}$
tangent:	$\tan \theta = \frac{y}{x}$	cotangent:	$\cot \theta = \frac{x}{y}$

These extended definitions agree with the right-triangle definitions when the angle is acute. Notice also that whenever the quotients are defined,

$$\begin{aligned}\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} \\ \sec \theta &= \frac{1}{\cos \theta} & \csc \theta &= \frac{1}{\sin \theta}\end{aligned}$$

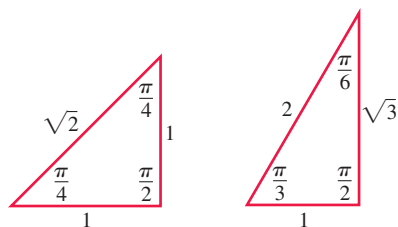


FIGURE 1.41 Radian angles and side lengths of two common triangles.

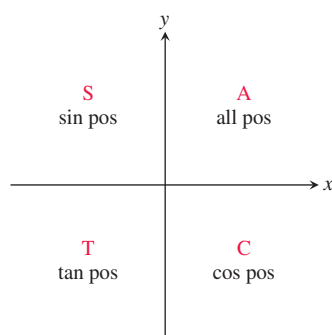


FIGURE 1.42 The ASTC rule, remembered by the statement “All Students Take Calculus,” tells which trigonometric functions are positive in each quadrant.

As you can see, $\tan \theta$ and $\sec \theta$ are not defined if $x = \cos \theta = 0$. This means they are not defined if θ is $\pm\pi/2, \pm3\pi/2, \dots$. Similarly, $\cot \theta$ and $\csc \theta$ are not defined for values of θ for which $y = 0$, namely $\theta = 0, \pm\pi, \pm2\pi, \dots$.

The exact values of these trigonometric ratios for some angles can be read from the triangles in Figure 1.41. For instance,

$$\begin{array}{lll} \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} & \sin \frac{\pi}{6} = \frac{1}{2} & \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \\ \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} & \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} & \cos \frac{\pi}{3} = \frac{1}{2} \\ \tan \frac{\pi}{4} = 1 & \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} & \tan \frac{\pi}{3} = \sqrt{3} \end{array}$$

The ASTC rule (Figure 1.42) is useful for remembering when the basic trigonometric functions are positive or negative. For instance, from the triangle in Figure 1.43, we see that

$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos \frac{2\pi}{3} = -\frac{1}{2}, \quad \tan \frac{2\pi}{3} = -\sqrt{3}.$$

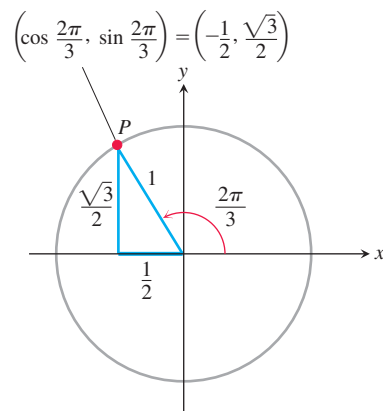


FIGURE 1.43 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

Using a similar method we obtain the values of $\sin \theta$, $\cos \theta$, and $\tan \theta$ shown in Table 1.2.

TABLE 1.2 Values of $\sin \theta$, $\cos \theta$, and $\tan \theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin \theta$	0	$-\frac{\sqrt{2}}{2}$	−1	$-\frac{\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos \theta$	−1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	0	1
$\tan \theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$-\frac{\sqrt{3}}{3}$	0		0

Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values: $\sin(\theta + 2\pi) = \sin \theta$, $\tan(\theta + 2\pi) = \tan \theta$, and so on. Similarly, $\cos(\theta - 2\pi) = \cos \theta$, $\sin(\theta - 2\pi) = \sin \theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic*.

Periods of Trigonometric Functions

Period π : $\tan(x + \pi) = \tan x$
 $\cot(x + \pi) = \cot x$

Period 2π : $\sin(x + 2\pi) = \sin x$
 $\cos(x + 2\pi) = \cos x$
 $\sec(x + 2\pi) = \sec x$
 $\csc(x + 2\pi) = \csc x$

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x . The smallest such value of p is the **period** of f .

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ . Figure 1.44 shows that the tangent and cotangent functions have period $p = \pi$, and the other four functions have period 2π . Also, the symmetries in these graphs reveal that the cosine and secant functions are even and the other four functions are odd (although this does not prove those results).

Even

$$\cos(-x) = \cos x$$

$$\sec(-x) = \sec x$$

Odd

$$\sin(-x) = -\sin x$$

$$\tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x$$

$$\cot(-x) = -\cot x$$

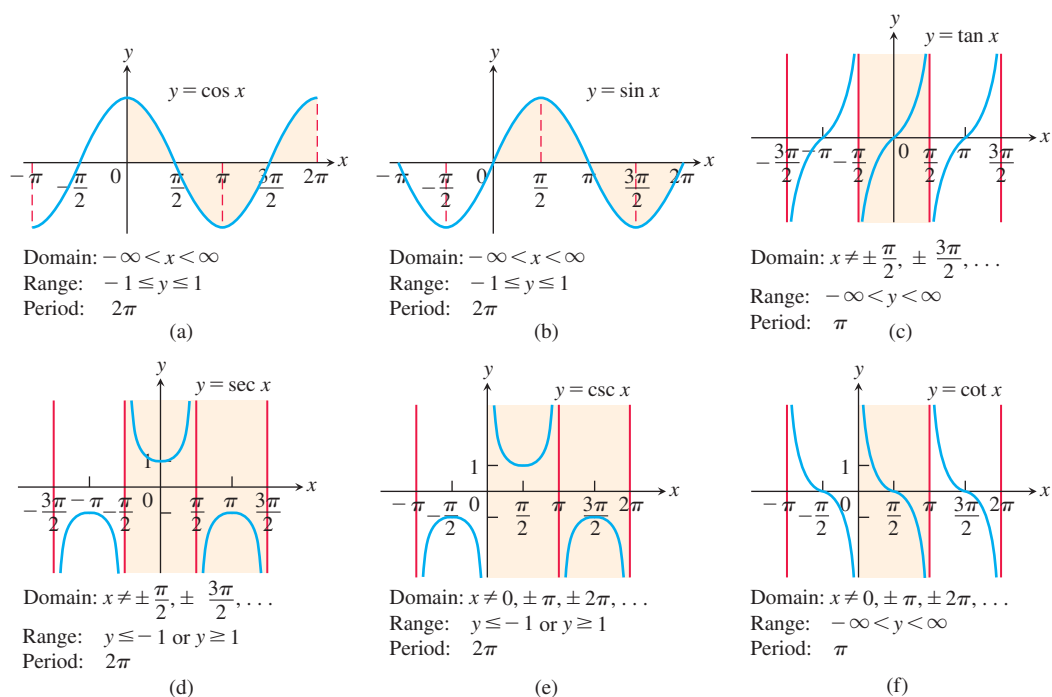


FIGURE 1.44 Graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.

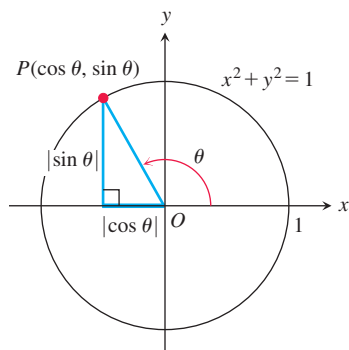


FIGURE 1.45 The reference triangle for a general angle θ .

Trigonometric Identities

The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance r from the origin and the angle θ that ray OP makes with the positive x -axis (Figure 1.40). Since $x/r = \cos \theta$ and $y/r = \sin \theta$, we have

$$x = r \cos \theta, \quad y = r \sin \theta.$$

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in Figure 1.45 and obtain the equation

$$\cos^2 \theta + \sin^2 \theta = 1. \quad (3)$$

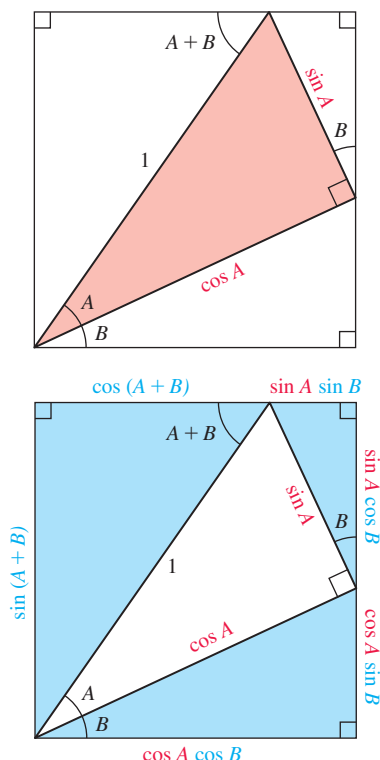


FIGURE 1.46 In a geometric proof of the angle sum identities we compare the opposite sides of the rectangle, which are equal. This assumes that A , B , and $A + B$ are acute, but the identities hold for all values of A and B .

This equation, true for all values of θ , is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2 \theta$ and $\sin^2 \theta$ gives

$$\begin{aligned} 1 + \tan^2 \theta &= \sec^2 \theta \\ 1 + \cot^2 \theta &= \csc^2 \theta \end{aligned}$$

The following formulas hold for all angles A and B (see Figure 1.46 and Exercise 58).

Addition Formulas

$$\begin{aligned} \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \sin(A + B) &= \sin A \cos B + \cos A \sin B \end{aligned} \quad (4)$$

There are similar formulas for $\cos(A - B)$ and $\sin(A - B)$ (Exercises 35 and 36). All the trigonometric identities needed in this text derive from Equations (3) and (4). For example, substituting θ for both A and B in the addition formulas gives

Double-Angle Formulas

$$\begin{aligned} \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ \sin 2\theta &= 2 \sin \theta \cos \theta \end{aligned} \quad (5)$$

Additional formulas come from combining the equations

$$\cos^2 \theta + \sin^2 \theta = 1, \quad \cos^2 \theta - \sin^2 \theta = \cos 2\theta.$$

We add the two equations to get $2 \cos^2 \theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2 \sin^2 \theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

Half-Angle Formulas

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad (6)$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad (7)$$

The Law of Cosines

If a , b , and c are sides of a triangle ABC and if θ is the angle opposite c , then

$$c^2 = a^2 + b^2 - 2ab \cos \theta. \quad (8)$$

This equation is called the **law of cosines**.

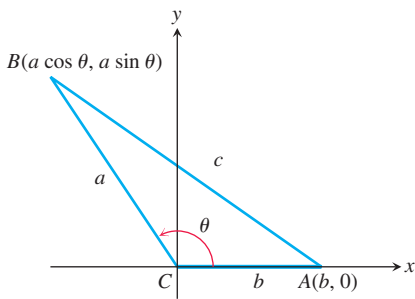


FIGURE 1.47 The square of the distance between A and B gives the law of cosines.

To see why the law holds, we position the triangle in the xy -plane with the origin at C and the positive x -axis along one side of the triangle, as in Figure 1.47. The coordinates of A are $(b, 0)$; the coordinates of B are $(a \cos \theta, a \sin \theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a \cos \theta - b)^2 + (a \sin \theta)^2 \\ &= a^2(\underbrace{\cos^2 \theta + \sin^2 \theta}_1) + b^2 - 2ab \cos \theta \\ &= a^2 + b^2 - 2ab \cos \theta. \end{aligned}$$

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos \theta = 0$ and $c^2 = a^2 + b^2$.

Two Special Inequalities

For any angle θ measured in radians, the sine and cosine functions satisfy

$$-|\theta| \leq \sin \theta \leq |\theta| \quad \text{and} \quad -|\theta| \leq 1 - \cos \theta \leq |\theta|.$$

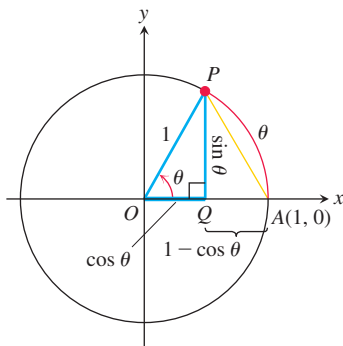


FIGURE 1.48 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2 \theta + (1 - \cos \theta)^2 \leq \theta^2$.

To establish these inequalities, we picture θ as a nonzero angle in standard position (Figure 1.48). The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc AP . The length of line segment AP is therefore less than $|\theta|$.

Triangle APQ is a right triangle with sides of length

$$QP = |\sin \theta|, \quad AQ = 1 - \cos \theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2 \theta + (1 - \cos \theta)^2 = (AP)^2 \leq \theta^2. \quad (9)$$

The terms on the left-hand side of Equation (9) are both positive, so each is smaller than their sum and hence is less than or equal to θ^2 :

$$\sin^2 \theta \leq \theta^2 \quad \text{and} \quad (1 - \cos \theta)^2 \leq \theta^2.$$

By taking square roots, this is equivalent to saying that

$$|\sin \theta| \leq |\theta| \quad \text{and} \quad |1 - \cos \theta| \leq |\theta|,$$

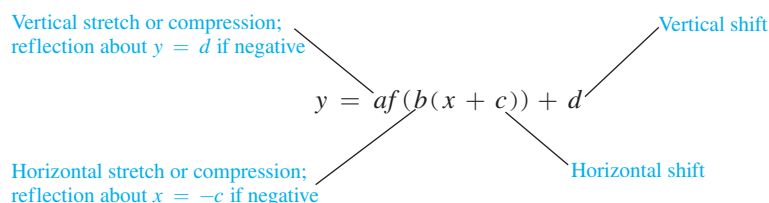
so

$$-|\theta| \leq \sin \theta \leq |\theta| \quad \text{and} \quad -|\theta| \leq 1 - \cos \theta \leq |\theta|.$$

These inequalities will be useful in the next chapter.

Transformations of Trigonometric Graphs

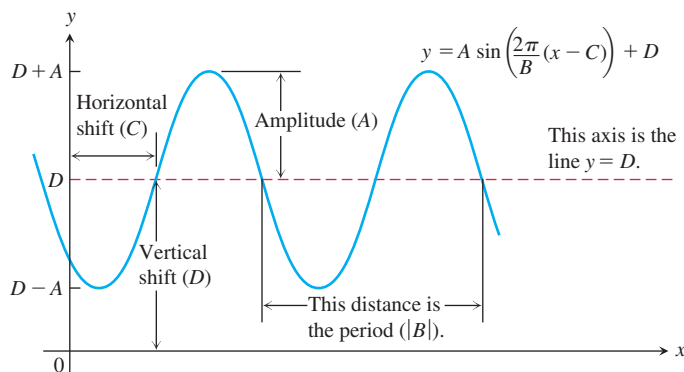
The rules for shifting, stretching, compressing, and reflecting the graph of a function summarized in the following diagram apply to the trigonometric functions we have discussed in this section.



The transformation rules applied to the sine function give the **general sine function** or **sinusoid** formula

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

where $|A|$ is the *amplitude*, $|B|$ is the *period*, C is the *horizontal shift*, and D is the *vertical shift*. A graphical interpretation of the various terms is given below.



EXERCISES 1.3

Radians and Degrees

- On a circle of radius 10 m, how long is an arc that subtends a central angle of (a) $4\pi/5$ radians? (b) 110° ?
- A central angle in a circle of radius 8 is subtended by an arc of length 10π . Find the angle's radian and degree measures.
- T** You want to make an 80° angle by marking an arc on the perimeter of a 12-cm-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest millimeter, how long should the arc be?
- T** If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

Evaluating Trigonometric Functions

- Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin \theta$					
$\cos \theta$					
$\tan \theta$					
$\cot \theta$					
$\sec \theta$					
$\csc \theta$					

- Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin \theta$					
$\cos \theta$					
$\tan \theta$					
$\cot \theta$					
$\sec \theta$					
$\csc \theta$					

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

- $\sin x = \frac{3}{5}$, $x \in \left[\frac{\pi}{2}, \pi\right]$
- $\tan x = 2$, $x \in \left[0, \frac{\pi}{2}\right]$
- $\cos x = \frac{1}{3}$, $x \in \left[-\frac{\pi}{2}, 0\right]$
- $\cos x = -\frac{5}{13}$, $x \in \left[\frac{\pi}{2}, \pi\right]$
- $\tan x = \frac{1}{2}$, $x \in \left[\pi, \frac{3\pi}{2}\right]$
- $\sin x = -\frac{1}{2}$, $x \in \left[\pi, \frac{3\pi}{2}\right]$

Graphing Trigonometric Functions

Graph the functions in Exercises 13–22. What is the period of each function?

- $\sin 2x$
- $\sin(x/2)$
- $\cos \pi x$
- $\cos \frac{\pi x}{2}$
- $-\sin \frac{\pi x}{3}$
- $-\cos 2\pi x$
- $\cos\left(x - \frac{\pi}{2}\right)$
- $\sin\left(x + \frac{\pi}{6}\right)$

$$21. \sin\left(x - \frac{\pi}{4}\right) + 1 \quad 22. \cos\left(x + \frac{2\pi}{3}\right) - 2$$

Graph the functions in Exercises 23–26 in the ts -plane (t -axis horizontal, s -axis vertical). What is the period of each function? What symmetries do the graphs have?

$$23. s = \cot 2t \quad 24. s = -\tan \pi t$$

$$25. s = \sec\left(\frac{\pi t}{2}\right) \quad 26. s = \csc\left(\frac{t}{2}\right)$$

- T** 27. a. Graph $y = \cos x$ and $y = \sec x$ together for $-\pi/2 \leq x \leq 3\pi/2$. Comment on the behavior of $\sec x$ in relation to the signs and values of $\cos x$.
 b. Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \leq x \leq 2\pi$. Comment on the behavior of $\csc x$ in relation to the signs and values of $\sin x$.
- T** 28. Graph $y = \tan x$ and $y = \cot x$ together for $-7 \leq x \leq 7$. Comment on the behavior of $\cot x$ in relation to the signs and values of $\tan x$.
29. Graph $y = \sin x$ and $y = |\sin x|$ together. What are the domain and range of $|\sin x|$?
30. Graph $y = \sin x$ and $y = \lceil \sin x \rceil$ together. What are the domain and range of $\lceil \sin x \rceil$?

Using the Addition Formulas

Use the addition formulas to derive the identities in Exercises 31–36.

$$31. \cos\left(x - \frac{\pi}{2}\right) = \sin x \quad 32. \cos\left(x + \frac{\pi}{2}\right) = -\sin x$$

$$33. \sin\left(x + \frac{\pi}{2}\right) = \cos x \quad 34. \sin\left(x - \frac{\pi}{2}\right) = -\cos x$$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$ (Exercise 57 provides a different derivation.)
36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$
37. What happens if you take $B = A$ in the trigonometric identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$? Does the result agree with something you already know?
38. What happens if you take $B = 2\pi$ in the addition formulas? Do the results agree with something you already know?

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

$$39. \cos(\pi + x) \quad 40. \sin(2\pi - x)$$

$$41. \sin\left(\frac{3\pi}{2} - x\right) \quad 42. \cos\left(\frac{3\pi}{2} + x\right)$$

$$43. \text{Evaluate } \sin \frac{7\pi}{12} \text{ as } \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right).$$

$$44. \text{Evaluate } \cos \frac{11\pi}{12} \text{ as } \cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right).$$

$$45. \text{Evaluate } \cos \frac{\pi}{12}. \quad 46. \text{Evaluate } \sin \frac{5\pi}{12}.$$

Using the Half-Angle Formulas

Find the function values in Exercises 47–50.

$$47. \cos^2 \frac{\pi}{8} \quad 48. \cos^2 \frac{5\pi}{12}$$

$$49. \sin^2 \frac{\pi}{12} \quad 50. \sin^2 \frac{3\pi}{8}$$

Solving Trigonometric Equations

For Exercises 51–54, solve for the angle θ , where $0 \leq \theta \leq 2\pi$.

$$51. \sin^2 \theta = \frac{3}{4} \quad 52. \sin^2 \theta = \cos^2 \theta$$

$$53. \sin 2\theta - \cos \theta = 0 \quad 54. \cos 2\theta + \cos \theta = 0$$

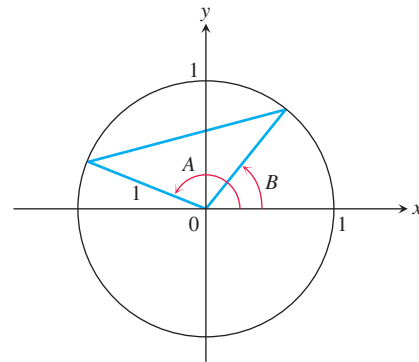
Theory and Examples

55. **The tangent sum formula** The standard formula for the tangent of the sum of two angles is

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the formula.

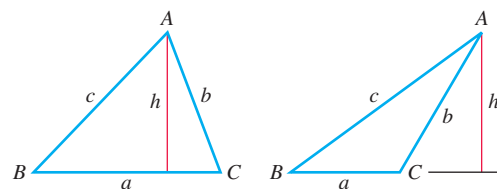
56. (Continuation of Exercise 55.) Derive a formula for $\tan(A - B)$.
57. Apply the law of cosines to the triangle in the accompanying figure to derive the formula for $\cos(A - B)$.



58. a. Apply the formula for $\cos(A - B)$ to the identity $\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$ to obtain the addition formula for $\sin(A + B)$.
 b. Derive the formula for $\cos(A + B)$ by substituting $-B$ for B in the formula for $\cos(A - B)$ from Exercise 35.
59. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60^\circ$. Find the length of side c .
60. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40^\circ$. Find the length of side c .
61. **The law of sines** The law of sines says that if a , b , and c are the sides opposite the angles A , B , and C in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Use the accompanying figures and the identity $\sin(\pi - \theta) = \sin \theta$, if required, to derive the law.

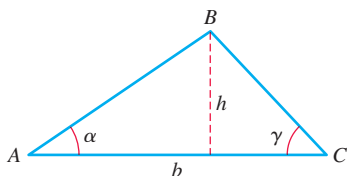


62. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60^\circ$ (as in Exercise 59). Find the sine of angle B using the law of sines.

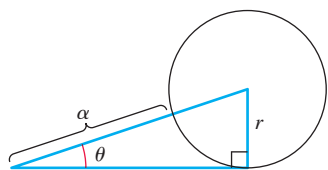
T 63. A triangle has side $c = 2$ and angles $A = \pi/4$ and $B = \pi/3$. Find the length a of the side opposite A .

64. Consider the length h of the perpendicular from point B to side b in the given triangle. Show that

$$h = \frac{b \tan \alpha \tan \gamma}{\tan \alpha + \tan \gamma}$$



65. Refer to the given figure. Write the radius r of the circle in terms of α and θ .



T 66. **The approximation $\sin x \approx x$** It is often useful to know that, when x is measured in radians, $\sin x \approx x$ for numerically small values of x . In Section 3.11, we will see why the approximation holds. The approximation error is less than 1 in 5000 if $|x| < 0.1$.

- With your grapher in radian mode, graph $y = \sin x$ and $y = x$ together in a viewing window about the origin. What do you see happening as x nears the origin?
- With your grapher in degree mode, graph $y = \sin x$ and $y = x$ together about the origin again. How is the picture different from the one obtained with radian mode?

General Sine Curves

For

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

identify A , B , C , and D for the sine functions in Exercises 67–70 and sketch their graphs.

- $y = 2 \sin(x + \pi) - 1$
- $y = \frac{1}{2} \sin(\pi x - \pi) + \frac{1}{2}$
- $y = -\frac{2}{\pi} \sin\left(\frac{\pi}{2}t\right) + \frac{1}{\pi}$
- $y = \frac{L}{2\pi} \sin \frac{2\pi t}{L}, \quad L > 0$

COMPUTER EXPLORATIONS

In Exercises 71–74, you will explore graphically the general sine function

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

as you change the values of the constants A , B , C , and D . Use a CAS or computer grapher to perform the steps in the exercises.

- The period B** Set the constants $A = 3$ and $C = D = 0$.
 - Plot $f(x)$ for the values $B = 1, 3, 2\pi, 5\pi$ over the interval $-4\pi \leq x \leq 4\pi$. Describe what happens to the graph of the general sine function as the period increases.
 - What happens to the graph for negative values of B ? Try it with $B = -3$ and $B = -2\pi$.
- The horizontal shift C** Set the constants $A = 3, B = 6, D = 0$.
 - Plot $f(x)$ for the values $C = 0, 1$, and 2 over the interval $-4\pi \leq x \leq 4\pi$. Describe what happens to the graph of the general sine function as C increases through positive values.
 - What happens to the graph for negative values of C ?
 - What smallest positive value should be assigned to C so the graph exhibits no horizontal shift? Confirm your answer with a plot.
- The vertical shift D** Set the constants $A = 3, B = 6, C = 0$.
 - Plot $f(x)$ for the values $D = 0, 1$, and 3 over the interval $-4\pi \leq x \leq 4\pi$. Describe what happens to the graph of the general sine function as D increases through positive values.
 - What happens to the graph for negative values of D ?
- The amplitude A** Set the constants $B = 6$ and $C = D = 0$.
 - Describe what happens to the graph of the general sine function as A increases through positive values. Confirm your answer by plotting $f(x)$ for the values $A = 1, 5$, and 9 .
 - What happens to the graph for negative values of A ?

1.4 Exponential Functions

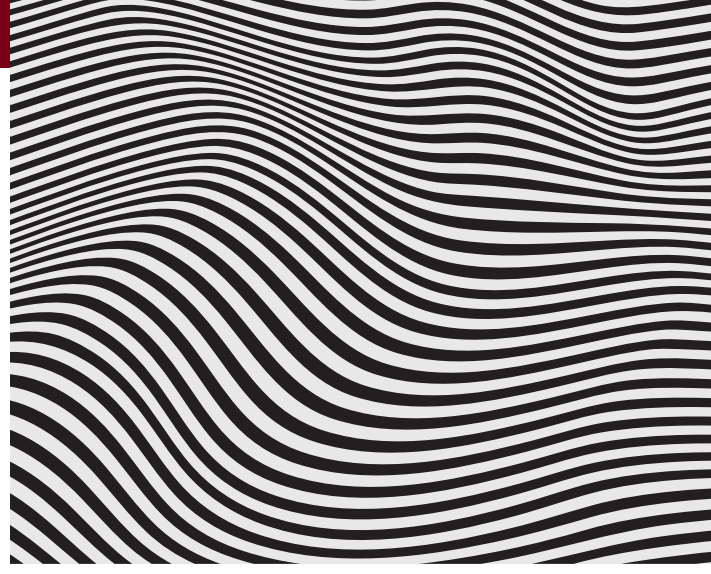
Exponential functions occur in a wide variety of applications, including interest rates, radioactive decay, population growth, the spread of a disease, consumption of natural resources, the earth's atmospheric pressure, temperature change of a heated object placed in a cooler environment, and the dating of fossils. In this section we introduce these functions informally, using an intuitive approach. We give a rigorous development of them in Chapter 7, based on the ideas of integral calculus.

Exponential Behavior

When a positive quantity P doubles, it increases by a factor of 2 and the quantity becomes $2P$. If it doubles again, it becomes $2(2P) = 2^2P$, and a third doubling gives $2(2^2P) = 2^3P$. Continuing to double in this fashion leads us to consider the function $f(x) = 2^x$. We call this an *exponential* function because the variable x appears in the exponent of 2^x . Functions

2

Limits and Continuity



OVERVIEW In this chapter we develop the concept of a limit, first intuitively and then formally. We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish among these behaviors.

2.1 Rates of Change and Tangent Lines to Curves

HISTORICAL BIOGRAPHY

Galileo Galilei
(1564–1642)

Galileo was an Italian mathematician and astronomer. He attempted to apply mathematics to his work in astronomy, physics of kinematics, and strength of materials.

To know more, visit the companion Website.

Average and Instantaneous Speed

In the late sixteenth century, Galileo discovered that a solid object dropped from rest (initially not moving) near the surface of the earth and allowed to fall freely, will fall a distance proportional to the square of the time it has been falling. This type of motion is called **free fall**. It assumes negligible air resistance to slow the object down, and it assumes that gravity is the only force acting on the falling object. If y denotes the distance fallen in meters after t seconds, then Galileo's law is

$$y = 4.9t^2 \text{ m,}$$

where 4.9 is the (approximate) constant of proportionality.

More generally, suppose that a moving object has traveled distance $f(t)$ at time t . The object's **average speed** during an interval of time $[t_1, t_2]$ is found by dividing the distance traveled $f(t_2) - f(t_1)$ by the time elapsed $t_2 - t_1$. The unit of measure is length per unit time: kilometers per hour, meters per second, or whatever is appropriate to the problem at hand.

Average Speed

When $f(t)$ measures the distance traveled at time t ,

$$\text{Average speed over } [t_1, t_2] = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

EXAMPLE 1 A rock breaks loose from the top of a tall cliff. What is its average speed

- (a) during the first 2 seconds of fall?
- (b) during the 1-second interval between second 1 and second 2?

Δ is the capital Greek letter Delta.

Solution The average speed of the rock during a given time interval is the change in distance, Δy , divided by the length of the time interval, Δt . (Increments like Δy and Δt are reviewed in Appendix A.4, and pronounced “delta y” and “delta t.”) Measuring distance in meters and time in seconds, we have the following calculations:

$$(a) \text{ For the first 2 seconds: } \frac{\Delta y}{\Delta t} = \frac{4.9(2)^2 - 4.9(0)^2}{2 - 0} = 9.8 \text{ m/s}$$

$$(b) \text{ From second 1 to second 2: } \frac{\Delta y}{\Delta t} = \frac{4.9(2)^2 - 4.9(1)^2}{2 - 1} = 14.7 \text{ m/s} \quad \blacksquare$$

We want a way to determine the speed of a falling object at a single instant t_0 , instead of using its average speed over an interval of time. To do this, we examine what happens when we calculate the average speed over shorter and shorter time intervals starting at t_0 . The next example illustrates this process. Our discussion is informal here but will be made precise in Chapter 3.

EXAMPLE 2 Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ s.

Solution We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{4.9(t_0 + h)^2 - 4.9t_0^2}{h}. \quad (1)$$

We cannot use this formula to calculate the “instantaneous” speed at the exact moment t_0 by simply substituting $h = 0$, because we cannot divide by zero. But we *can* use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$. When we do so, by taking smaller and smaller values of h , we see a pattern (Table 2.1).

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

Average speed: $\frac{\Delta y}{\Delta t} = \frac{4.9(t_0 + h)^2 - 4.9t_0^2}{h}$		
Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	14.7	24.5
0.1	10.29	20.09
0.01	9.849	19.649
0.001	9.8049	19.6049
0.0001	9.80049	19.60049

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 9.8 as the length of the interval decreases. This suggests that the rock is falling at a speed of 9.8 m/s at $t_0 = 1$ s. Let’s confirm this algebraically.

If we set $t_0 = 1$ and then expand the numerator in Equation (1) and simplify, we find that

$$\begin{aligned}\frac{\Delta y}{\Delta t} &= \frac{4.9(1+h)^2 - 4.9(1)^2}{h} = \frac{4.9(1+2h+h^2) - 4.9}{h} \\ &= \frac{9.8h + 4.9h^2}{h} = 9.8 + 4.9h. \quad \text{Can cancel } h \text{ when } h \neq 0\end{aligned}$$

For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $9.8 + 4.9h$ m/sec. We can now see why the average speed has the limiting value $9.8 + 4.9(0) = 9.8$ m/sec as h approaches 0.

Similarly, setting $t_0 = 2$ in Equation (1), for values of h different from 0 the procedure yields

$$\frac{\Delta y}{\Delta t} = 19.6 + 4.9h.$$

As h gets closer and closer to 0, the average speed has the limiting value 19.6 m/s when $t_0 = 2$ s, as suggested by Table 2.1. ■

The average speed of a falling object is an example of a more general idea, an average rate of change.

Average Rates of Change and Secant Lines

Given any function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y , $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs. (We use the symbol h for Δx to simplify the notation here and later on.)

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0.$$

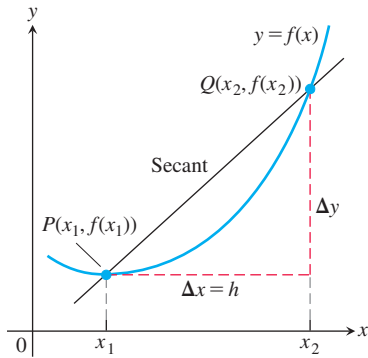


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Figure 2.1). In geometry, a line joining two points of a curve is called a **secant line**. Thus, the average rate of change of f from x_1 to x_2 is identical to the slope of secant line PQ . As the point Q approaches the point P along the curve, the length h of the interval over which the change occurs approaches zero. We will see that this procedure leads to the definition of the slope of a curve at a point.

Defining the Slope of a Curve

We know what is meant by the slope of a straight line, which tells us the rate at which it rises or falls—its rate of change as a linear function. But what is meant by the *slope of a curve* at a point P on the curve? If there were a *tangent line* to the curve at P —a line that grazes the curve like the tangent line to a circle—it would be reasonable to identify the slope of the tangent line as the slope of the curve at P . We will see that, among all the lines that pass through the point P , the tangent line is the one that gives the best approximation to the curve at P . We need a precise way to specify the tangent line at a point on a curve.

Specifying a tangent line to a circle is straightforward. A line L is tangent to a circle at a point P if L passes through P and is perpendicular to the radius at P (Figure 2.2). But what does it mean to say that a line L is tangent to a more general curve at a point P ?

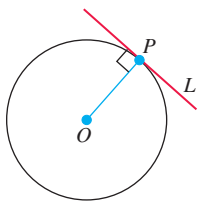


FIGURE 2.2 L is tangent to the circle at P if it passes through P perpendicular to radius OP .

HISTORICAL BIOGRAPHY**Pierre de Fermat
(1601–1665)**

Fermat was born to a prosperous family in France. He studied the classics and mastered Latin, Greek, Italian, and Spanish.

To know more, visit the companion Website.

To define tangency for general curves, we use an approach that analyzes the behavior of the secant lines that pass through P and nearby points Q as Q moves toward P along the curve (Figure 2.3). We start with what we *can* calculate, namely the slope of the secant line PQ . We then compute the limiting value of the secant line's slope as Q approaches P along the curve. (We clarify the limit idea in the next section.) If the limit exists, we take it to be the slope of the curve at P and *define* the tangent line to the curve at P to be the line through P with this slope.

The next example illustrates the geometric idea for finding the tangent line to a curve.

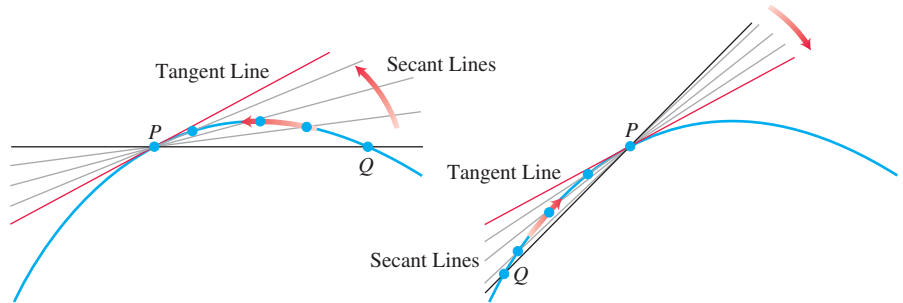


FIGURE 2.3 The tangent line to the curve at P is the line through P whose slope is the limit of the secant line slopes as $Q \rightarrow P$ from either side.

EXAMPLE 3 Find the slope of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$ by analyzing the slopes of secant lines through $(2, 4)$. Write an equation for the tangent line to the parabola at this point.

Solution We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$, as shown in Figure 2.4. We then write an expression for the slope of the secant line PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\begin{aligned} \text{Secant line slope} &= \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h} \\ &= \frac{h^2 + 4h}{h} = h + 4. \end{aligned}$$

If $h > 0$, then Q lies above and to the right of P , as in Figure 2.4. If $h < 0$, then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant line slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at P .

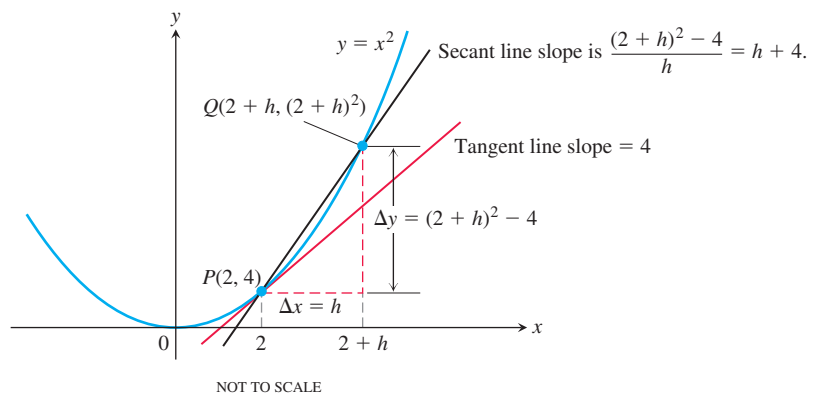


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant line slopes (Example 3).

The tangent line to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \quad \text{Point-slope equation}$$

$$y = 4x - 4. \quad \text{Simplify.}$$

Rates of Change and Tangent Lines

The rates at which the rock in Example 2 was falling at the instants $t = 1$ and $t = 2$ are called *instantaneous rates of change*. Instantaneous rates of change and slopes of tangent lines are closely connected, as we see in the following examples.

EXAMPLE 4 Figure 2.5 shows how a population p of fruit flies (*Drosophila*) grew in a 50-day experiment. The flies were counted at regular intervals, the counted values plotted with respect to the number of elapsed days t , and the points joined by a smooth curve (colored blue in Figure 2.5). Find the average growth rate from day 23 to day 45.

Solution There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by $340 - 150 = 190$ in $45 - 23 = 22$ days. The average rate of change of the population from day 23 to day 45 was

$$\frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day.}$$

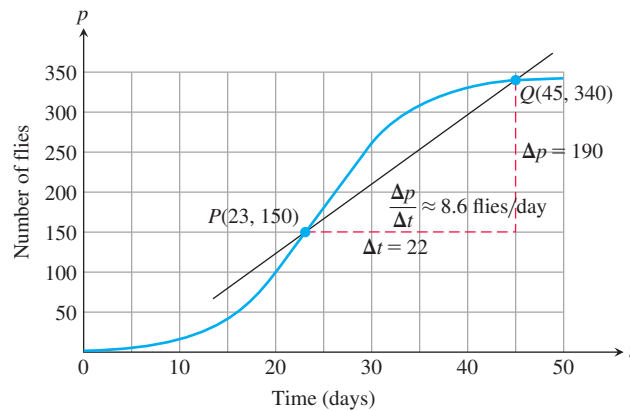


FIGURE 2.5 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line (Example 4).

This average is the slope of the secant line through the points P and Q on the graph in Figure 2.5.

The average rate of change from day 23 to day 45 calculated in Example 4 does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.

EXAMPLE 5 How fast was the number of flies in the population of Example 4 growing on day 23?

Solution To answer this question, we examine the average rates of change over shorter and shorter time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secant lines from P to Q , for a sequence of points Q approaching P along the curve (Figure 2.6).

Q	Slope of $PQ = \Delta p / \Delta t$ (flies/day)
$(45, 340)$	$\frac{340 - 150}{45 - 23} \approx 8.6$
$(40, 330)$	$\frac{330 - 150}{40 - 23} \approx 10.6$
$(35, 310)$	$\frac{310 - 150}{35 - 23} \approx 13.3$
$(30, 265)$	$\frac{265 - 150}{30 - 23} \approx 16.4$

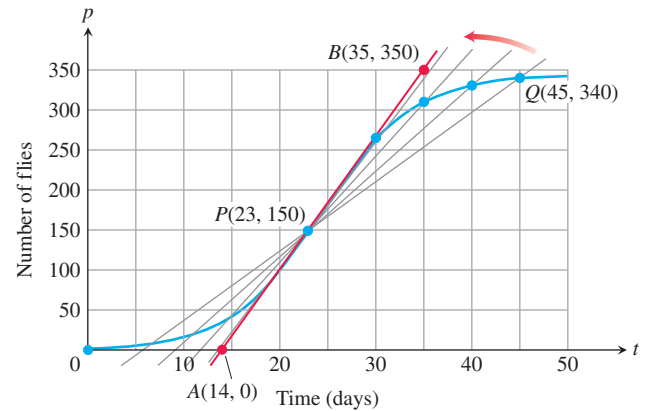


FIGURE 2.6 The positions and slopes of four secant lines through the point P on the fruit fly graph (Example 5).

The values in the table show that the secant line slopes rise from 8.6 to 16.4 as the t -coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued decreasing toward 23. Geometrically, the secant lines rotate counter-clockwise about P and seem to approach the red tangent line in the figure. Since the line appears to pass through the points $(14, 0)$ and $(35, 350)$, its slope is approximately

$$\frac{350 - 0}{35 - 14} = 16.7 \text{ flies/day.}$$

On day 23 the population was increasing at a rate of about 16.7 flies/day. ■

The instantaneous rate of change is the value the average rate of change approaches as the length h of the interval over which the change occurs approaches zero. The average rate of change corresponds to the slope of a secant line; the instantaneous rate corresponds to the slope of the tangent line at a fixed value. So instantaneous rates and slopes of tangent lines are closely connected. We give a precise definition for these terms in the next chapter, but to do so we first need to develop the concept of a *limit*.

EXERCISES 2.1

Average Rates of Change

In Exercises 1–6, find the average rate of change of the function over the given interval or intervals.

- $f(x) = x^3 + 1$
 - $[2, 3]$
 - $[-1, 1]$
- $g(x) = x^2 - 2x$
 - $[1, 3]$
 - $[-2, 4]$
- $h(t) = \cot t$
 - $[\pi/4, 3\pi/4]$
 - $[\pi/6, \pi/2]$
- $g(t) = 2 + \cos t$
 - $[0, \pi]$
 - $[-\pi, \pi]$
- $R(\theta) = \sqrt{4\theta + 1}$; $[0, 2]$
- $P(\theta) = \theta^3 - 4\theta^2 + 5\theta$; $[1, 2]$

Slope of a Curve at a Point

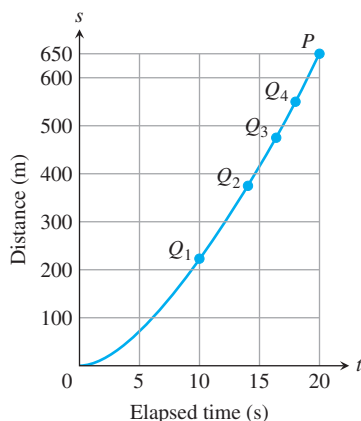
In Exercises 7–18, use the method in Example 3 to find (a) the slope of the curve at the given point P , and (b) an equation of the tangent line at P .

- $y = x^2 - 5$, $P(2, -1)$
- $y = 7 - x^2$, $P(2, 3)$
- $y = x^2 - 2x - 3$, $P(2, -3)$
- $y = x^2 - 4x$, $P(1, -3)$
- $y = x^3$, $P(2, 8)$
- $y = 2 - x^3$, $P(1, 1)$
- $y = x^3 - 12x$, $P(1, -11)$
- $y = x^3 - 3x^2 + 4$, $P(2, 0)$
- $y = \frac{1}{x}$, $P(-2, -1/2)$

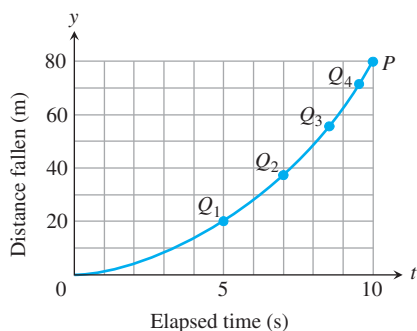
16. $y = \frac{x}{2-x}$, $P(4, -2)$
 17. $y = \sqrt{x}$, $P(4, 2)$
 18. $y = \sqrt{7-x}$, $P(-2, 3)$

Instantaneous Rates of Change

- T 19. Speed of a car** The accompanying figure shows the time-to-distance graph for a sports car accelerating from a standstill.



- a. Estimate the slopes of secant lines PQ_1 , PQ_2 , PQ_3 , and PQ_4 , arranging them in order in a table like the one in Figure 2.6. What are the appropriate units for these slopes?
 b. Then estimate the car's speed at time $t = 20$ s.
- T 20.** The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.
- a. Estimate the slopes of the secant lines PQ_1 , PQ_2 , PQ_3 , and PQ_4 , arranging them in a table like the one in Figure 2.6.
 b. About how fast was the object going when it hit the surface?



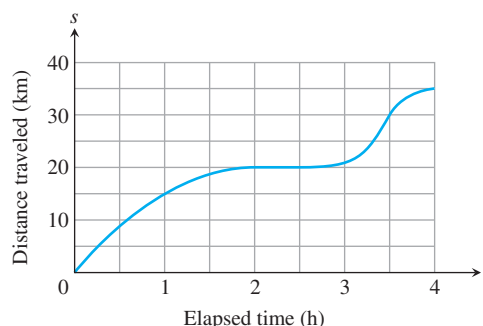
- T 21.** The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in \$1000s
2017	6
2018	27
2019	62
2020	111
2021	174

- a. Plot points representing the profit as a function of year, and join them by as smooth a curve as you can.

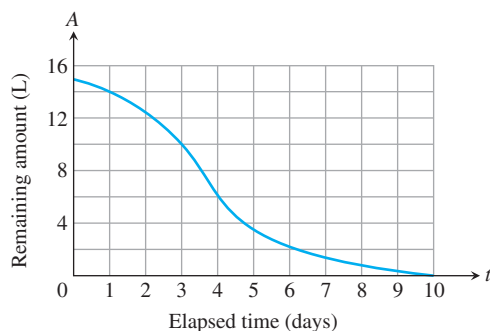
- b. What is the average rate of increase of the profits between 2019 and 2021?
 c. Use your graph to estimate the rate at which the profits were changing in 2019.
- T 22.** Make a table of values for the function $F(x) = (x+2)/(x-2)$ at the points $x = 1.2$, $x = 11/10$, $x = 101/100$, $x = 1001/1000$, $x = 10001/10000$, and $x = 1$.
- a. Find the average rate of change of $F(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in your table.
 b. Extending the table if necessary, try to determine the rate of change of $F(x)$ at $x = 1$.
- T 23.** Let $g(x) = \sqrt{x}$ for $x \geq 0$.
- a. Find the average rate of change of $g(x)$ with respect to x over the intervals $[1, 2]$, $[1, 1.5]$ and $[1, 1+h]$.
 b. Make a table of values of the average rate of change of g with respect to x over the interval $[1, 1+h]$ for some values of h approaching zero, say $h = 0.1, 0.01, 0.001, 0.0001, 0.00001$, and 0.000001 .
 c. What does your table indicate is the rate of change of $g(x)$ with respect to x at $x = 1$?
- T 24.** Let $f(t) = 1/t$ for $t \neq 0$.
- a. Find the average rate of change of f with respect to t over the intervals (i) from $t = 2$ to $t = 3$, and (ii) from $t = 2$ to $t = T$.
 b. Make a table of values of the average rate of change of f with respect to t over the interval $[2, T]$, for some values of T approaching 2, say $T = 2.1, 2.01, 2.001, 2.0001, 2.00001$, and 2.000001 .
 c. What does your table indicate is the rate of change of f with respect to t at $t = 2$?

25. The accompanying graph shows the total distance s traveled by a bicyclist after t hours.



- a. Estimate the bicyclist's average speed over the time intervals $[0, 1]$, $[1, 2.5]$, and $[2.5, 3.5]$.
 b. Estimate the bicyclist's instantaneous speed at the times $t = \frac{1}{2}$, $t = 2$, and $t = 3$.
 c. Estimate the bicyclist's maximum speed and the specific time at which it occurs.

26. The accompanying graph shows the total amount of gasoline A in the gas tank of a motorcycle after being driven for t days.



- Estimate the average rate of gasoline consumption over the time intervals $[0, 3]$, $[0, 5]$, and $[7, 10]$.
- Estimate the instantaneous rate of gasoline consumption at the times $t = 1$, $t = 4$, and $t = 8$.
- Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

2.2 Limit of a Function and Limit Laws

HISTORICAL ESSAY

Limits

To read this essay, visit the companion Website.

In Section 2.1 we saw how limits arise when finding the instantaneous rate of change of a function or the tangent line to a curve. We begin this section by presenting an informal definition of the limit of a function. We then describe laws that capture the behavior of limits. These laws enable us to quickly compute limits for a variety of functions, including polynomials and rational functions. We will present the precise definition of a limit in Section 2.3.

Limits of Function Values

Frequently, when studying a function $y = f(x)$, we find ourselves interested in the function's behavior *near* a particular point c , but not *at* c itself. An important example occurs when the process of trying to evaluate a function at c leads to division by zero, which is undefined. We encountered this when seeking the instantaneous rate of change in y by considering the quotient function $\Delta y/h$ for h closer and closer to zero. In the next example we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (since we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \quad \text{for } x \neq 1.$$

The graph of f is the line $y = x + 1$ with the point $(1, 2)$ *removed*. This removed point is shown as a “hole” in Figure 2.7. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ as close as we want to 2 by choosing x close enough to 1 (Table 2.2).

We will illustrate some other types of behavior near a point in Example 3.

An Informal Description of the Limit of a Function

We now give an informal definition of the limit of a function f at an interior point of the domain of f . Suppose that $f(x)$ is defined on an open interval about c , *except possibly at* c

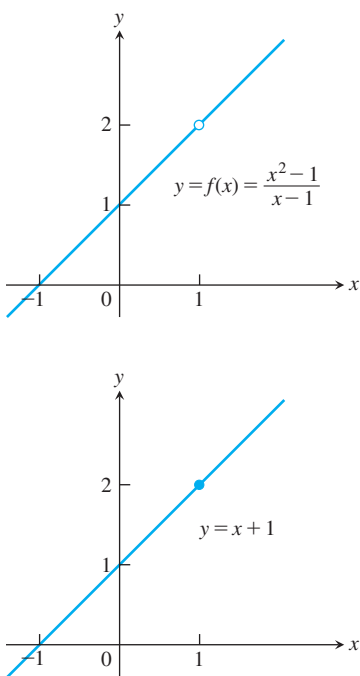


FIGURE 2.7 The graph of f is identical to the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \frac{x^2 - 1}{x - 1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

itself. If $f(x)$ is arbitrarily close to the number L (that is, as close to L as we like) for all x sufficiently close to c , other than c itself, then we say that f approaches the **limit** L as x approaches c , and write

$$\lim_{x \rightarrow c} f(x) = L.$$

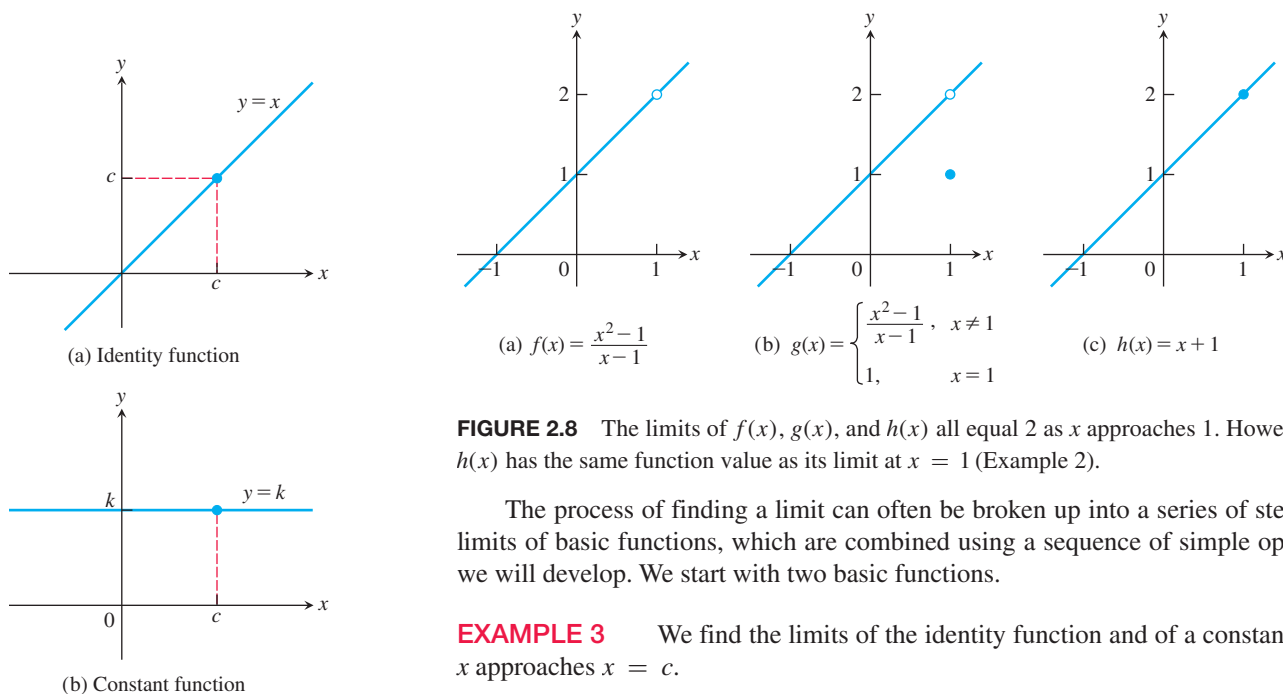
This is read “the limit of $f(x)$ as x approaches c is L .” In Example 1 we would say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \rightarrow 1} f(x) = 2, \quad \text{or} \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2.$$

Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to c . The value of the function at c itself is not considered.

Our definition here is informal, because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. (To a machinist manufacturing a piston, *close* may mean *within a few hundredths of a millimeter*. To an astronomer studying distant galaxies, *close* may mean *within a few thousand light-years*.) Nevertheless, the definition is clear enough to enable us to recognize and evaluate limits of many specific functions. We will need the precise definition given in Section 2.3 when we set out to prove theorems about limits or study complicated functions. Here are several more examples exploring the idea of limits.

EXAMPLE 2 The limit of a function does not depend on how the function is defined at the point being approached. It does not even matter whether the function is defined at that point. Consider the three functions in Figure 2.8. The function f has limit 2 as $x \rightarrow 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \rightarrow 1$ even though $2 \neq g(1)$. The function h is the only one of the three functions in Figure 2.8 whose limit as $x \rightarrow 1$ equals its value at $x = 1$. For h , we have $\lim_{x \rightarrow 1} h(x) = h(1)$. This equality of limit and function value has an important meaning. As illustrated by the three examples in Figure 2.8, equality of limit and function value captures the notion of “continuity.” We study this in detail in Section 2.6. ■

**FIGURE 2.9** The functions in Example 3 have limits at all points c .**FIGURE 2.8** The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

The process of finding a limit can often be broken up into a series of steps involving limits of basic functions, which are combined using a sequence of simple operations that we will develop. We start with two basic functions.

EXAMPLE 3 We find the limits of the identity function and of a constant function as x approaches $x = c$.

(a) If f is the **identity function** $f(x) = x$, then for any value of c (Figure 2.9a),

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x = c.$$

- (b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of c (Figure 2.9b),

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} k = k.$$

For instances of each of these rules we have

$$\lim_{x \rightarrow 3} x = 3 \quad \text{Limit of identity function at } x = 3$$

and

$$\lim_{x \rightarrow -7} 4 = \lim_{x \rightarrow 2} 4 = 4. \quad \begin{array}{l} \text{Limit of constant function} \\ f(x) = 4 \text{ at } x = -7 \text{ or at } x = 2 \end{array}$$

We prove these rules in Example 3 in Section 2.3. ■

A function may not have a limit at a particular point. Some ways that limits can fail to exist are illustrated in Figure 2.10 and described in the next example.

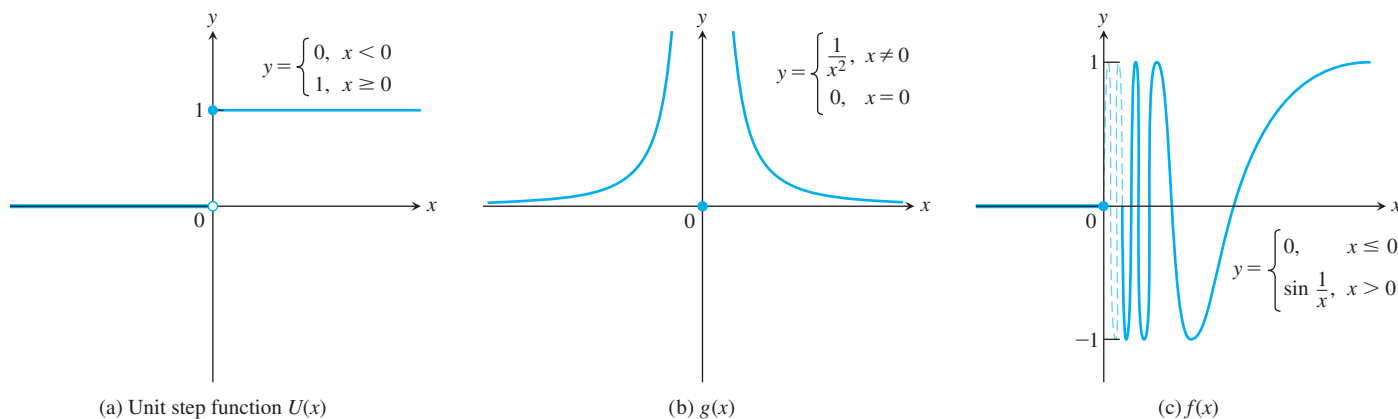


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

EXAMPLE 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \rightarrow 0$.

- (a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$
- (b) $g(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
- (c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$

Solution

- (a) This function *jumps*: The **unit step function** $U(x)$ has no limit as $x \rightarrow 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \rightarrow 0$ (Figure 2.10a).
- (b) This function *grows too “large” to have a limit*: $g(x)$ has no limit as $x \rightarrow 0$ because the values of g grow arbitrarily large as $x \rightarrow 0$ and therefore do not stay close to *any* fixed real number (Figure 2.10b). We say that the function g is *not bounded*.

The function U in Example 4 is known as the *Heaviside function*, after Oliver Heaviside (1850–1925).

- (c) This function *oscillates too much to have a limit*: $f(x)$ has no limit as $x \rightarrow 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0 . The values do not stay close to any single number as $x \rightarrow 0$ (Figure 2.10c). ■

A function that oscillates may or may not have a limit. In Example 11 we will see a function that oscillates wildly, but nevertheless does have a limit. The problem with the function f discussed in Example 4 is not that it oscillates, but that it oscillates *too much* for a limit to exist.

The Limit Laws

A few basic rules allow us to break down complicated functions into simple ones when calculating limits. By using these laws, we can greatly simplify many limit computations.

THEOREM 1—Limit Laws

If L , M , c , and k are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

- | | |
|------------------------------------|---|
| 1. <i>Sum Rule</i> : | $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$ |
| 2. <i>Difference Rule</i> : | $\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$ |
| 3. <i>Constant Multiple Rule</i> : | $\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$ |
| 4. <i>Product Rule</i> : | $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$ |
| 5. <i>Quotient Rule</i> : | $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$ |
| 6. <i>Power Rule</i> : | $\lim_{x \rightarrow c} [f(x)]^n = L^n, \quad n \text{ a positive integer}$ |
| 7. <i>Root Rule</i> : | $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n \text{ a positive integer}$ |

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c .)

The Sum Rule says that the limit of a sum is the sum of the limits. Similarly, the next rules say that the limit of a difference is the difference of the limits; the limit of a constant times a function is the constant times the limit of the function; the limit of a product is the product of the limits; the limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0); the limit of a positive integer power (or root) of a function is the integer power (or root) of the limit (provided that the root of the limit is a real number).

There are simple intuitive arguments for why the properties in Theorem 1 are true (although these do not constitute proofs). If x is sufficiently close to c , then $f(x)$ is close to L and $g(x)$ is close to M , from our informal definition of a limit. It is then reasonable that $f(x) + g(x)$ is close to $L + M$; $f(x) - g(x)$ is close to $L - M$; $kf(x)$ is close to kL ; $f(x)g(x)$ is close to LM ; and $f(x)/g(x)$ is close to L/M if M is not zero. We prove the Sum Rule in Section 2.3, based on a rigorous definition of the limit. Rules 2–5 are proved in Appendix A.6. Rule 6 is obtained by applying Rule 4 repeatedly. Rule 7 is proved in more advanced texts. The Sum, Difference, and Product Rules can be extended to any number of functions, not just two.

EXAMPLE 5 Use the observations $\lim_{x \rightarrow c} k = k$ and $\lim_{x \rightarrow c} x = c$ (Example 3) and the limit laws in Theorem 1 to find the following limits.

(a) $\lim_{x \rightarrow c} (x^3 + 4x^2 - 3)$ (b) $\lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5}$ (c) $\lim_{x \rightarrow -2} \sqrt{4x^2 + 3}$

Solution

$$\begin{aligned}
 \text{(a)} \quad \lim_{x \rightarrow c} (x^3 + 4x^2 - 3) &= \lim_{x \rightarrow c} x^3 + \lim_{x \rightarrow c} 4x^2 - \lim_{x \rightarrow c} 3 && \text{Sum and Difference Rules} \\
 &= c^3 + 4c^2 - 3 && \text{Power and Multiple Rules and limit of a constant function} \\
 \\
 \text{(b)} \quad \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} &= \frac{\lim_{x \rightarrow c} (x^4 + x^2 - 1)}{\lim_{x \rightarrow c} (x^2 + 5)} && \text{Quotient Rule: Note that } (x^2 + 5) > 0 \text{ for all } x. \\
 &= \frac{\lim_{x \rightarrow c} x^4 + \lim_{x \rightarrow c} x^2 - \lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} x^2 + \lim_{x \rightarrow c} 5} && \text{Sum and Difference Rules} \\
 &= \frac{c^4 + c^2 - 1}{c^2 + 5} && \text{Power Rule and limit of a constant function} \\
 \\
 \text{(c)} \quad \lim_{x \rightarrow -2} \sqrt{4x^2 + 3} &= \sqrt{\lim_{x \rightarrow -2} (4x^2 + 3)} && \text{Root Rule with } n = 2 (4x^2 + 3 \geq 0) \\
 &= \sqrt{\lim_{x \rightarrow -2} 4x^2 + \lim_{x \rightarrow -2} 3} && \text{Difference Rule} \\
 &= \sqrt{4(-2)^2 + 3} && \text{Power and Multiple Rules and limit of a constant function} \\
 &= \sqrt{16 + 3} \\
 &= \sqrt{19}
 \end{aligned}$$

Evaluating Limits of Polynomials and Rational Functions

Theorem 1 simplifies the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c , just substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point c at which the denominator is not zero, substitute c for x in the formula for the function. (See Examples 5a and 5b.) We state these results formally as theorems.

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 6 The following calculation illustrates Theorems 2 and 3:

$$\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Since the denominator of this rational expression does not equal 0 when we substitute -1 for x , we can just compute the value of the expression at $x = -1$ to evaluate the limit. ■

Eliminating Common Factors from Zero Denominators

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c . If the denominator is zero, canceling common factors in the numerator and

Identifying Common Factors

If $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

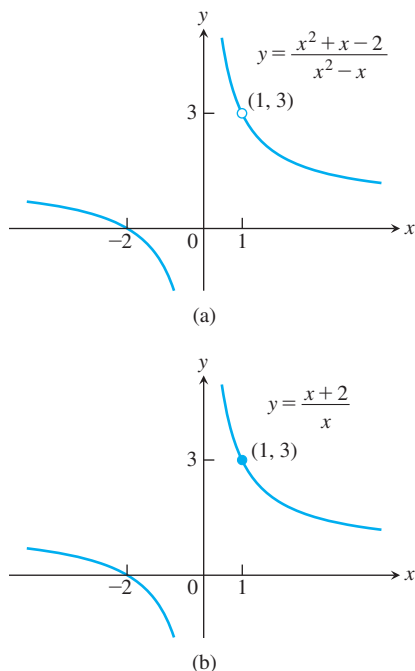


FIGURE 2.11 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \rightarrow 1$ (Example 7).

denominator may reduce the fraction to one whose denominator is no longer zero at c . If this happens, we can find the limit by substitution in the simplified fraction.

EXAMPLE 7 Evaluate

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \quad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \rightarrow 1$ by evaluating the function at $x = 1$, as in Theorem 3:

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \rightarrow 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

See Figure 2.11. ■

Using Calculators and Computers to Estimate Limits

We can try using a calculator or computer to guess a limit numerically. However, calculators and computers can sometimes give false values and misleading evidence about limits. Usually the problem is associated with rounding errors, as we now illustrate.

EXAMPLE 8 Estimate the value of $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$.

Solution Table 2.3 lists values of the function obtained on a calculator for several points approaching $x = 0$. As x approaches 0 through the points $\pm 1, \pm 0.5, \pm 0.1$, and ± 0.01 , the function seems to approach the number 0.05.

As we take even smaller values of x , $\pm 0.0005, \pm 0.0001, \pm 0.00001$, and ± 0.000001 , the function appears to approach the number 0.

Is the answer 0.05 or 0, or some other value? We resolve this question in the next example. ■

Using a computer or calculator may give ambiguous results, as in Example 8. A computer cannot always keep track of enough digits to avoid rounding errors in computing the values of $f(x)$ when x is very small. We cannot substitute $x = 0$ in the problem, and the

TABLE 2.3 Computed values of $f(x) = \frac{\sqrt{x^2 + 100} - 10}{x^2}$ near $x = 0$

x	$f(x)$	
± 1	0.049876	approaches 0.05?
± 0.5	0.049969	
± 0.1	0.049999	
± 0.01	0.050000	
± 0.0005	0.050000	approaches 0?
± 0.0001	0.000000	
± 0.00001	0.000000	
± 0.000001	0.000000	

numerator and denominator have no obvious common factors (as they did in Example 7). Sometimes, however, we can create a common factor by using algebra.

EXAMPLE 9 Evaluate

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

Solution This is the limit we considered in Example 8. We can create a common factor by multiplying both numerator and denominator by the conjugate radical expression $\sqrt{x^2 + 100} + 10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\begin{aligned} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} && \text{Multiply and divide by the conjugate.} \\ &= \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} && \text{Simplify.} \\ &= \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)} && \text{Common factor } x^2 \\ &= \frac{1}{\sqrt{x^2 + 100} + 10}. && \text{Cancel } x^2 \text{ for } x \neq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 100} + 10} \\ &= \frac{1}{\sqrt{0^2 + 100} + 10} && \text{Use limit laws: Sum Rule, Power Rule, Root Rule, and Quotient Rule (denominator not 0).} \\ &= \frac{1}{20} = 0.05. \end{aligned}$$

This calculation provides the correct answer, resolving the ambiguous computer results in Example 8. ■

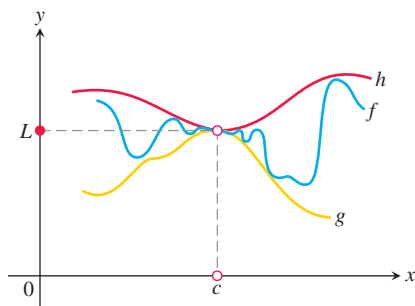


FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h .

We cannot always manipulate the terms in an expression to find the limit of a quotient where the denominator becomes zero. In some cases the limit might then be found with geometric arguments (see the proof of Theorem 6 in Section 2.4), or through methods of calculus (developed in Section 4.5). The next theorem shows how to evaluate difficult limits by comparing them with functions having known limits.

The Sandwich Theorem

The following theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c . Being trapped between the values of two functions that approach L , the values of f must also approach L (Figure 2.12). A proof is given in Appendix A.6.

THEOREM 4—The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c , except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L.$$

Then $\lim_{x \rightarrow c} f(x) = L$.

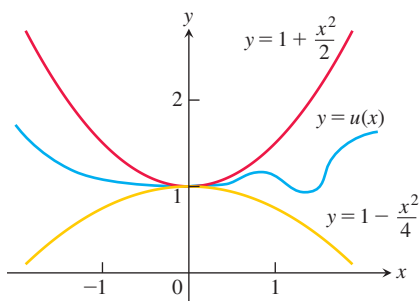


FIGURE 2.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \rightarrow 0$ (Example 10).

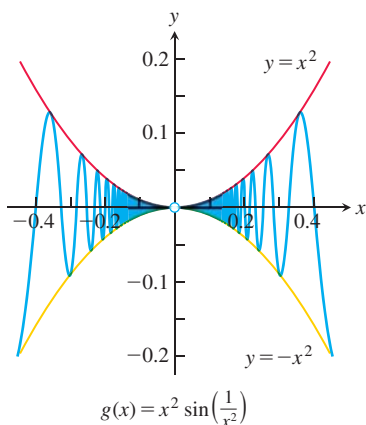


FIGURE 2.14 The graph of the function g (Example 11). It is not defined at $x = 0$. Even though the function oscillates, it has a limit as $x \rightarrow 0$. The value of $g(x)$ always lies between x^2 and $-x^2$.

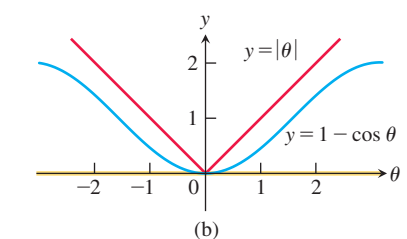
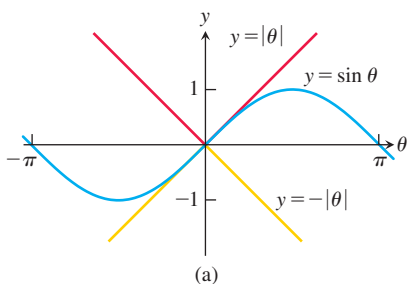


FIGURE 2.15 The Sandwich Theorem confirms the limits in Example 12.

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.

EXAMPLE 10 Given a function u that satisfies

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \quad \text{for all } x \neq 0,$$

find $\lim_{x \rightarrow 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x \rightarrow 0} (1 - (x^2/4)) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} (1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x \rightarrow 0} u(x) = 1$ (Figure 2.13). ■

We use the Sandwich Theorem to show that it is possible for a function that oscillates to have a limit.

EXAMPLE 11 How does the function $g(x) = x^2 \sin(1/x^2)$ behave near $x = 0$?

Solution The formula defines $g(x)$ for all real numbers x except $x = 0$. The graph of g is shown in Figure 2.14. We can see that the graph oscillates, but we can use the Sandwich Theorem to find the limit of $g(x)$ as x approaches 0. If $x \neq 0$, then $1/x^2$ is a positive real number. Since the range of the sine function is the interval $[-1, 1]$, it follows that $-1 \leq \sin(1/x^2) \leq 1$ for all $x \neq 0$. Even though we may not know the exact value of $g(x) = x^2 \sin(1/x^2)$, we do know that it lies between $-x^2$ and x^2 . Since

$$\lim_{x \rightarrow 0} x^2 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (-x^2) = 0,$$

the Sandwich Theorem implies that $\lim_{x \rightarrow 0} g(x) = 0$. ■

EXAMPLE 12 The Sandwich Theorem helps us establish several important limit rules:

- (a) $\lim_{\theta \rightarrow 0} \sin \theta = 0$
- (b) $\lim_{\theta \rightarrow 0} \cos \theta = 1$
- (c) For any function f , $\lim_{x \rightarrow c} |f(x)| = 0$ implies $\lim_{x \rightarrow c} f(x) = 0$.

Solution

- (a) In Section 1.3 we established that $-|\theta| \leq \sin \theta \leq |\theta|$ for all θ (see Figure 2.15a). Since $\lim_{\theta \rightarrow 0} (-|\theta|) = \lim_{\theta \rightarrow 0} |\theta| = 0$, we have

$$\lim_{\theta \rightarrow 0} \sin \theta = 0.$$

- (b) From Section 1.3, $0 \leq 1 - \cos \theta \leq |\theta|$ for all θ (see Figure 2.15b). Since $\lim_{\theta \rightarrow 0} |\theta| = 0$ and $\lim_{\theta \rightarrow 0} 0 = 0$, we have $\lim_{\theta \rightarrow 0} (1 - \cos \theta) = 0$ so

$$\lim_{\theta \rightarrow 0} \cos \theta = \lim_{\theta \rightarrow 0} (1 - (1 - \cos \theta)) = 1 - \lim_{\theta \rightarrow 0} (1 - \cos \theta) = 1 - 0 = 1.$$

- (c) Since $-|f(x)| \leq f(x) \leq |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \rightarrow c$, it follows that $\lim_{x \rightarrow c} f(x) = 0$. ■

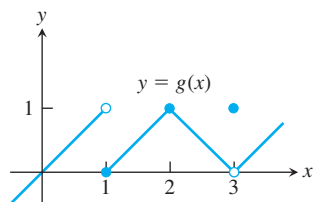
Example 12 shows that the sine and cosine functions are equal to their limits at $\theta = 0$. We have not yet established that for any c , $\lim_{\theta \rightarrow c} \sin \theta = \sin c$, and $\lim_{\theta \rightarrow c} \cos \theta = \cos c$. These limit formulas do hold, as will be shown in Section 2.6.

EXERCISES 2.2

Limits from Graphs

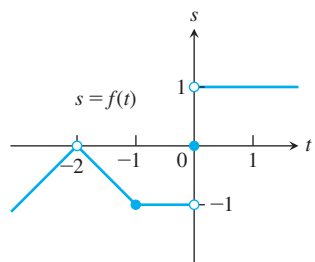
1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

a. $\lim_{x \rightarrow 1} g(x)$ b. $\lim_{x \rightarrow 2} g(x)$ c. $\lim_{x \rightarrow 3} g(x)$ d. $\lim_{x \rightarrow 2.5} g(x)$



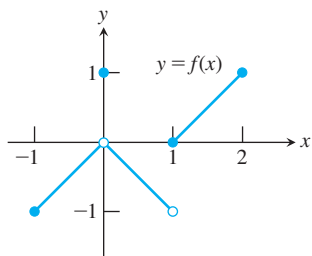
2. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

a. $\lim_{t \rightarrow -2} f(t)$ b. $\lim_{t \rightarrow -1} f(t)$ c. $\lim_{t \rightarrow 0} f(t)$ d. $\lim_{t \rightarrow 0.5} f(t)$



3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

- a. $\lim_{x \rightarrow 0} f(x)$ exists.
 b. $\lim_{x \rightarrow 0} f(x) = 0$
 c. $\lim_{x \rightarrow 0} f(x) = 1$
 d. $\lim_{x \rightarrow 1} f(x) = 1$
 e. $\lim_{x \rightarrow 1} f(x) = 0$
 f. $\lim_{x \rightarrow c} f(x)$ exists at every point c in $(-1, 1)$.
 g. $\lim_{x \rightarrow 1} f(x)$ does not exist.
 h. $f(0) = 0$
 i. $f(0) = 1$
 j. $f(1) = 0$
 k. $f(1) = -1$

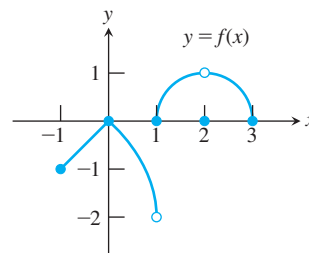


4. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

- a. $\lim_{x \rightarrow 2} f(x)$ does not exist.
 b. $\lim_{x \rightarrow 2} f(x) = 2$
 c. $\lim_{x \rightarrow 1} f(x)$ does not exist.
 d. $\lim_{x \rightarrow c} f(x)$ exists at every point c in $(-1, 1)$.

- e. $\lim_{x \rightarrow c} f(x)$ exists at every point c in $(1, 3)$.

- f. $f(1) = 0$
 g. $f(1) = -2$
 h. $f(2) = 0$
 i. $f(2) = 1$



Existence of Limits

In Exercises 5 and 6, explain why the limits do not exist.

5. $\lim_{x \rightarrow 0} \frac{x}{|x|}$

6. $\lim_{x \rightarrow 1} \frac{1}{x - 1}$

7. Suppose that a function $f(x)$ is defined for all real values of x except $x = c$. Can anything be said about the existence of $\lim_{x \rightarrow c} f(x)$? Give reasons for your answer.

8. Suppose that a function $f(x)$ is defined for all x in $[-1, 1]$. Can anything be said about the existence of $\lim_{x \rightarrow 0} f(x)$? Give reasons for your answer.

9. If $\lim_{x \rightarrow 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

10. If $f(1) = 5$, must $\lim_{x \rightarrow 1} f(x)$ exist? If it does, then must $\lim_{x \rightarrow 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x \rightarrow 1} f(x)$? Explain.

Calculating Limits

Find the limits in Exercises 11–22.

11. $\lim_{x \rightarrow -3} (x^2 - 13)$

12. $\lim_{x \rightarrow 2} (-x^2 + 5x - 2)$

13. $\lim_{t \rightarrow 6} 8(t - 5)(t - 7)$

14. $\lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8)$

15. $\lim_{x \rightarrow 2} \frac{2x + 5}{11 - x^3}$

16. $\lim_{s \rightarrow 2/3} (8 - 3s)(2s - 1)$

17. $\lim_{x \rightarrow -1/2} 4x(3x + 4)^2$

18. $\lim_{y \rightarrow 2} \frac{y + 2}{y^2 + 5y + 6}$

19. $\lim_{y \rightarrow -3} (5 - y)^{4/3}$

20. $\lim_{z \rightarrow 4} \sqrt{z^2 - 10}$

21. $\lim_{h \rightarrow 0} \frac{3}{\sqrt{3h + 1} + 1}$

22. $\lim_{h \rightarrow 0} \frac{\sqrt{5h + 4} - 2}{h}$

Limits of quotients Find the limits in Exercises 23–42.

23. $\lim_{x \rightarrow 5} \frac{x - 5}{x^2 - 25}$

24. $\lim_{x \rightarrow -3} \frac{x + 3}{x^2 + 4x + 3}$

25. $\lim_{x \rightarrow -5} \frac{x^2 + 3x - 10}{x + 5}$

26. $\lim_{x \rightarrow 2} \frac{x^2 - 7x + 10}{x - 2}$

27. $\lim_{t \rightarrow 1} \frac{t^2 + t - 2}{t^2 - 1}$

28. $\lim_{t \rightarrow -1} \frac{t^2 + 3t + 2}{t^2 - t - 2}$

29. $\lim_{t \rightarrow -2} \frac{-2x - 4}{x^3 + 2x^2}$

30. $\lim_{y \rightarrow 0} \frac{5y^3 + 8y^2}{3y^4 - 16y^2}$

$$31. \lim_{x \rightarrow 1} \frac{x^{-1} - 1}{x - 1} \qquad 32. \lim_{x \rightarrow 0} \frac{\frac{1}{x-1} + \frac{1}{x+1}}{x}$$

$$33. \lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 - 1} \qquad 34. \lim_{v \rightarrow 2} \frac{v^3 - 8}{v^4 - 16}$$

$$35. \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \qquad 36. \lim_{x \rightarrow 4} \frac{4x - x^2}{2 - \sqrt{x}}$$

$$37. \lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x} + 3 - 2} \qquad 38. \lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$$

$$39. \lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 12} - 4}{x - 2} \qquad 40. \lim_{x \rightarrow -2} \frac{x + 2}{\sqrt{x^2 + 5} - 3}$$

$$41. \lim_{x \rightarrow -3} \frac{2 - \sqrt{x^2 - 5}}{x + 3} \qquad 42. \lim_{x \rightarrow 4} \frac{4 - x}{5 - \sqrt{x^2 + 9}}$$

Limits with trigonometric functions Find the limits in Exercises 43–50.

$$43. \lim_{x \rightarrow 0} (2 \sin x - 1) \qquad 44. \lim_{x \rightarrow 0} \sin^2 x$$

$$45. \lim_{x \rightarrow 0} \sec x \qquad 46. \lim_{x \rightarrow 0} \tan x$$

$$47. \lim_{x \rightarrow 0} \frac{1 + x + \sin x}{3 \cos x} \qquad 48. \lim_{x \rightarrow 0} (x^2 - 1)(2 - \cos x)$$

$$49. \lim_{x \rightarrow -\pi} \sqrt{x + 4} \cos(x + \pi) \qquad 50. \lim_{x \rightarrow 0} \sqrt{7 + \sec^2 x}$$

Using Limit Rules

51. Suppose $\lim_{x \rightarrow 0} f(x) = 1$ and $\lim_{x \rightarrow 0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x \rightarrow 0} \frac{2f(x) - g(x)}{(f(x) + 7)^2} = \frac{\lim_{x \rightarrow 0} (2f(x) - g(x))}{\lim_{x \rightarrow 0} (f(x) + 7)^2} \quad (a)$$

(We assume the denominator is nonzero.)

$$= \frac{\lim_{x \rightarrow 0} 2f(x) - \lim_{x \rightarrow 0} g(x)}{\left(\lim_{x \rightarrow 0} (f(x) + 7)\right)^2} \quad (b)$$

$$= \frac{2 \lim_{x \rightarrow 0} f(x) - \lim_{x \rightarrow 0} g(x)}{\left(\lim_{x \rightarrow 0} f(x) + \lim_{x \rightarrow 0} 7\right)^2} \quad (c)$$

$$= \frac{(2)(1) - (-5)}{(1 + 7)^2} = \frac{7}{64}$$

52. Let $\lim_{x \rightarrow 1} h(x) = 5$, $\lim_{x \rightarrow 1} p(x) = 1$, and $\lim_{x \rightarrow 1} r(x) = 2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x \rightarrow 1} \frac{\sqrt{5h(x)}}{p(x)(4 - r(x))} = \frac{\lim_{x \rightarrow 1} \sqrt{5h(x)}}{\lim_{x \rightarrow 1} (p(x)(4 - r(x)))} \quad (a)$$

(We assume the denominator is nonzero.)

$$= \frac{\sqrt{\lim_{x \rightarrow 1} 5h(x)}}{\left(\lim_{x \rightarrow 1} p(x)\right)\left(\lim_{x \rightarrow 1} (4 - r(x))\right)} \quad (b)$$

$$= \frac{\sqrt{5 \lim_{x \rightarrow 1} h(x)}}{\left(\lim_{x \rightarrow 1} p(x)\right)\left(\lim_{x \rightarrow 1} 4 - \lim_{x \rightarrow 1} r(x)\right)} \quad (c)$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4 - 2)} = \frac{5}{2}$$

53. Suppose $\lim_{x \rightarrow c} f(x) = 5$ and $\lim_{x \rightarrow c} g(x) = -2$. Find

$$a. \lim_{x \rightarrow c} f(x)g(x) \qquad b. \lim_{x \rightarrow c} 2f(x)g(x)$$

$$c. \lim_{x \rightarrow c} (f(x) + 3g(x)) \qquad d. \lim_{x \rightarrow c} \frac{f(x)}{f(x) - g(x)}$$

54. Suppose $\lim_{x \rightarrow 4} f(x) = 0$ and $\lim_{x \rightarrow 4} g(x) = -3$. Find

$$a. \lim_{x \rightarrow 4} (g(x) + 3) \qquad b. \lim_{x \rightarrow 4} xf(x)$$

$$c. \lim_{x \rightarrow 4} (g(x))^2 \qquad d. \lim_{x \rightarrow 4} \frac{g(x)}{f(x) - 1}$$

55. Suppose $\lim_{x \rightarrow b} f(x) = 7$ and $\lim_{x \rightarrow b} g(x) = -3$. Find

$$a. \lim_{x \rightarrow b} (f(x) + g(x)) \qquad b. \lim_{x \rightarrow b} f(x) \cdot g(x)$$

$$c. \lim_{x \rightarrow b} 4g(x) \qquad d. \lim_{x \rightarrow b} f(x)/g(x)$$

56. Suppose that $\lim_{x \rightarrow -2} p(x) = 4$, $\lim_{x \rightarrow -2} r(x) = 0$, and

$$\lim_{x \rightarrow -2} s(x) = -3. \text{ Find}$$

$$a. \lim_{x \rightarrow -2} (p(x) + r(x) + s(x))$$

$$b. \lim_{x \rightarrow -2} (p(x) \cdot r(x) \cdot s(x))$$

$$c. \lim_{x \rightarrow -2} (-4p(x) + 5r(x))/s(x)$$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

occur frequently in calculus. In Exercises 57–62, evaluate this limit for the given value of x and function f .

$$57. f(x) = x^2, \quad x = 1$$

$$58. f(x) = x^2, \quad x = -2$$

$$59. f(x) = 3x - 4, \quad x = 2$$

$$60. f(x) = 1/x, \quad x = -2$$

$$61. f(x) = \sqrt{x}, \quad x = 7$$

$$62. f(x) = \sqrt{3x + 1}, \quad x = 0$$

Using the Sandwich Theorem

63. If $\sqrt{5 - 2x^2} \leq f(x) \leq \sqrt{5 - x^2}$ for $-1 \leq x \leq 1$, find $\lim_{x \rightarrow 0} f(x)$.

64. If $2 - x^2 \leq g(x) \leq 2 \cos x$ for all x , find $\lim_{x \rightarrow 0} g(x)$.

65. a. It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$$

hold for all values of x close to zero (except for $x = 0$). What, if anything, does this tell you about

$$\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}?$$

Give reasons for your answer.

T b. Graph $y = 1 - (x^2/6)$, $y = (x \sin x)/(2 - 2 \cos x)$, and $y = 1$ together for $-2 \leq x \leq 2$. Comment on the behavior of the graphs as $x \rightarrow 0$.

66. a. Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero, except for $x = 0$ itself. (They do, as you will see in Section 16.9.) What, if anything, does this tell you about

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

- T** b. Graph the equations $y = (1/2) - (x^2/24)$, $y = (1 - \cos x)/x^2$, and $y = 1/2$ together for $-2 \leq x \leq 2$. Comment on the behavior of the graphs as $x \rightarrow 0$.

Estimating Limits

T You will find a graphing calculator useful for Exercises 67–76.

67. Let $f(x) = (x^2 - 9)/(x + 3)$.
- Make a table of the values of f at the points $x = -3.1, -3.01, -3.001$, and so on as far as your calculator can go. Then estimate $\lim_{x \rightarrow -3} f(x)$. What estimate do you arrive at if you evaluate f at $x = -2.9, -2.99, -2.999, \dots$ instead?
 - Support your conclusions in part (a) by graphing f near $c = -3$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow -3$.
 - Find $\lim_{x \rightarrow -3} f(x)$ algebraically, as in Example 7.
68. Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.
- Make a table of the values of g at the points $x = 1.4, 1.41, 1.414$, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x \rightarrow \sqrt{2}} g(x)$.
 - Support your conclusion in part (a) by graphing g near $c = \sqrt{2}$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow \sqrt{2}$.
 - Find $\lim_{x \rightarrow \sqrt{2}} g(x)$ algebraically.
69. Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.
- Make a table of the values of G at $x = -5.9, -5.99, -5.999$, and so on. Then estimate $\lim_{x \rightarrow -6} G(x)$. What estimate do you arrive at if you evaluate G at $x = -6.1, -6.01, -6.001, \dots$ instead?
 - Support your conclusions in part (a) by graphing G and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow -6$.
 - Find $\lim_{x \rightarrow -6} G(x)$ algebraically.
70. Let $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$.
- Make a table of the values of h at $x = 2.9, 2.99, 2.999$, and so on. Then estimate $\lim_{x \rightarrow 3} h(x)$. What estimate do you arrive at if you evaluate h at $x = 3.1, 3.01, 3.001, \dots$ instead?
 - Support your conclusions in part (a) by graphing h near $c = 3$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow 3$.
 - Find $\lim_{x \rightarrow 3} h(x)$ algebraically.
71. Let $f(x) = (x^2 - 1)/(|x| - 1)$.
- Make tables of the values of f at values of x that approach $c = -1$ from above and below. Then estimate $\lim_{x \rightarrow -1} f(x)$.

- Support your conclusion in part (a) by graphing f near $c = -1$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow -1$.
 - Find $\lim_{x \rightarrow -1} f(x)$ algebraically.
72. Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.
- Make tables of values of F at values of x that approach $c = -2$ from above and below. Then estimate $\lim_{x \rightarrow -2} F(x)$.
 - Support your conclusion in part (a) by graphing F near $c = -2$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow -2$.
 - Find $\lim_{x \rightarrow -2} F(x)$ algebraically.
73. Let $g(\theta) = (\sin \theta)/\theta$.
- Make a table of the values of g at values of θ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta \rightarrow 0} g(\theta)$.
 - Support your conclusion in part (a) by graphing g near $\theta_0 = 0$.
74. Let $G(t) = (1 - \cos t)/t^2$.
- Make tables of values of G at values of t that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t \rightarrow 0} G(t)$.
 - Support your conclusion in part (a) by graphing G near $t_0 = 0$.
75. Let $f(x) = x^{1/(1-x)}$.
- Make tables of values of f at values of x that approach $c = 1$ from above and below. Does f appear to have a limit as $x \rightarrow 1$? If so, what is it? If not, why not?
 - Support your conclusions in part (a) by graphing f near $c = 1$.
76. Let $f(x) = (3^x - 1)/x$.
- Make tables of values of f at values of x that approach $c = 0$ from above and below. Does f appear to have a limit as $x \rightarrow 0$? If so, what is it? If not, why not?
 - Support your conclusions in part (a) by graphing f near $c = 0$.

Theory and Examples

77. If $x^4 \leq f(x) \leq x^2$ for x in $[-1, 1]$ and $x^2 \leq f(x) \leq x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x \rightarrow c} f(x)$? What can you say about the value of the limit at these points?
78. Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x \neq 2$ and suppose that
- $$\lim_{x \rightarrow 2} g(x) = \lim_{x \rightarrow 2} h(x) = -5.$$
- Can we conclude anything about the values of f , g , and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x \rightarrow 2} f(x) = 0$? Give reasons for your answers.
79. If $\lim_{x \rightarrow 4} \frac{f(x) - 5}{x - 2} = 1$, find $\lim_{x \rightarrow 4} f(x)$.
80. If $\lim_{x \rightarrow -2} \frac{f(x)}{x^2} = 1$, find
- $\lim_{x \rightarrow -2} f(x)$,
 - $\lim_{x \rightarrow -2} \frac{f(x)}{x}$.
81. a. If $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 3$, find $\lim_{x \rightarrow 2} f(x)$.
- If $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 4$, find $\lim_{x \rightarrow 2} f(x)$.

82. If $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 1$, find

- a. $\lim_{x \rightarrow 0} f(x)$, b. $\lim_{x \rightarrow 0} \frac{f(x)}{x}$.

T 83. a. Graph $g(x) = x \sin(1/x)$ to estimate $\lim_{x \rightarrow 0} g(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

T 84. a. Graph $h(x) = x^2 \cos(1/x^3)$ to estimate $\lim_{x \rightarrow 0} h(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

COMPUTER EXPLORATIONS

Graphical Estimates of Limits

In Exercises 85–90, use a CAS to perform the following steps:

- Plot the function near the point c being approached.
- From your plot, guess the value of the limit.

85. $\lim_{x \rightarrow 2} \frac{x^4 - 16}{x - 2}$

86. $\lim_{x \rightarrow -1} \frac{x^3 - x^2 - 5x - 3}{(x + 1)^2}$

87. $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - 1}{x}$

88. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{\sqrt{x^2 + 7} - 4}$

89. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sin x}$

90. $\lim_{x \rightarrow 0} \frac{2x^2}{3 - 3 \cos x}$

2.3 The Precise Definition of a Limit

We now turn our attention to the precise definition of a limit. The early history of calculus saw controversy about the validity of the basic concepts underlying the theory. Apparent contradictions were argued over by both mathematicians and philosophers. These controversies were resolved by the precise definition, which allows us to replace vague phrases like “gets arbitrarily close to” in the informal definition with specific conditions that can be applied to any particular example. With a rigorous definition, we can avoid misunderstandings, prove the limit properties given in the preceding section, and establish many important limits.

To show that the limit of $f(x)$ as $x \rightarrow c$ equals the number L , we need to show that the gap between $f(x)$ and L can be made “as small as we choose” if x is kept “close enough” to c . Let us see what this requires if we specify the size of the gap between $f(x)$ and L .

EXAMPLE 1 Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it seems clear that y is close to 7 when x is close to 4, so $\lim_{x \rightarrow 4} (2x - 1) = 7$. However, how close to $x = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x :

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: What values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

$$\begin{aligned} |2x - 8| &< 2 \\ -2 &< 2x - 8 < 2 && \text{Removing absolute value gives two inequalities.} \\ 6 &< 2x < 10 && \text{Add 8 to each term.} \\ 3 &< x < 5 && \text{Solve for } x. \\ -1 &< x - 4 < 1. && \text{Solve for } x - 4. \end{aligned}$$

Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Figure 2.16). ■

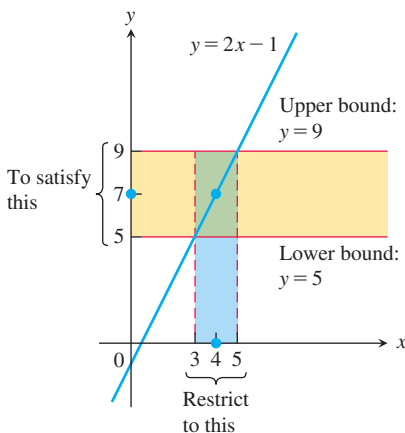


FIGURE 2.16 Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Example 1).

In the previous example we determined how close x must be to a particular value c to ensure that the outputs $f(x)$ of some function lie within a prescribed interval about a limit value L . To show that the limit of $f(x)$ as $x \rightarrow c$ actually equals L , we must be able to show that the gap between $f(x)$ and L can be made less than *any prescribed error*, no matter how

δ is the Greek letter delta.
 ε is the Greek letter epsilon.

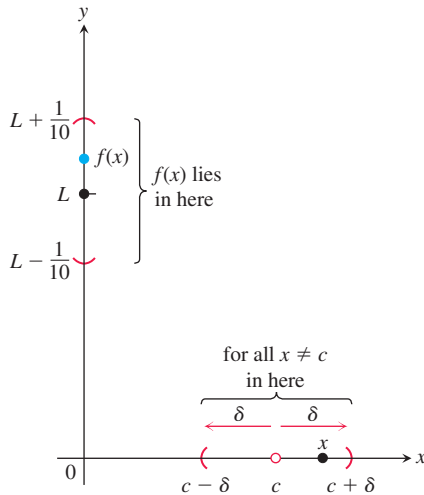


FIGURE 2.17 How should we define $\delta > 0$ so that keeping x within the interval $(c - \delta, c + \delta)$ will keep $f(x)$ within the interval $(L - \frac{1}{10}, L + \frac{1}{10})$?

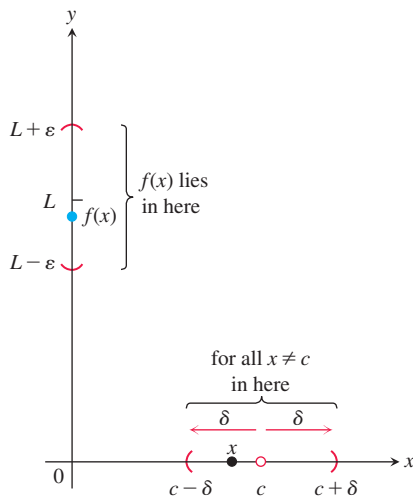


FIGURE 2.18 The relation of δ and ε in the definition of limit.

small, by holding x close enough to c . To describe arbitrary prescribed errors, we introduce two constants, δ (delta) and ε (epsilon). These Greek letters are traditionally used to represent small changes in a variable or a function.

Definition of Limit

Suppose we are watching the values of a function $f(x)$ as x approaches c (without taking on the value c itself). Certainly we want to be able to say that $f(x)$ stays within one-tenth of a unit from L as soon as x stays within some distance δ of c (Figure 2.17). But that in itself is not enough, because as x continues on its course toward c , what is to prevent $f(x)$ from jumping around within the interval from $L - (1/10)$ to $L + (1/10)$ without tending toward L ? We can be told that the error can be no more than $1/100$ or $1/1000$ or $1/100,000$. Each time, we find a new δ -interval about c so that keeping x within that interval satisfies the new error tolerance. And each time the possibility exists that $f(x)$ might jump away from L at some later stage.

The figures on the next page illustrate the problem. You can think of this as a quarrel between a skeptic and a scholar. The skeptic presents ε -challenges to show there is room for doubt that the limit exists. The scholar counters every challenge with a δ -interval around c which ensures that the function takes values within ε of L .

How do we stop this seemingly endless series of challenges and responses? We can do so by proving that for *every* error tolerance ε that the challenger can produce, we can present a matching distance δ that keeps x “close enough” to c to keep $f(x)$ within that ε -tolerance of L (Figure 2.18). This leads us to the precise definition of a limit.

DEFINITION Let $f(x)$ be defined on an open interval about c , except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L** , and write

$$\lim_{x \rightarrow c} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

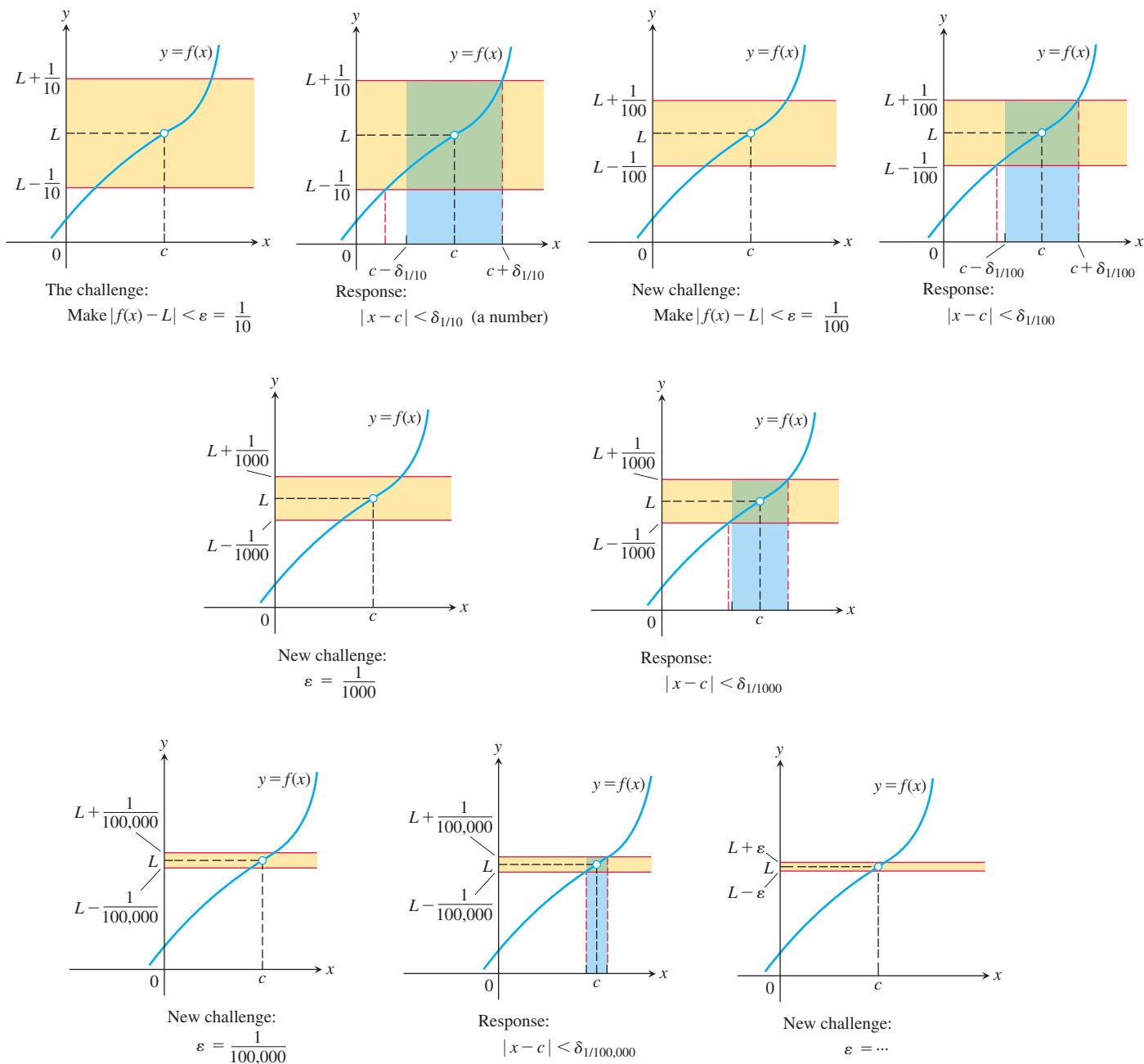
$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

To visualize the definition, imagine machining a cylindrical shaft to a close tolerance. The diameter of the shaft is determined by turning a dial to a setting measured by a variable x . We try for diameter L , but since nothing is perfect we must be satisfied with a diameter $f(x)$ somewhere between $L - \varepsilon$ and $L + \varepsilon$. The number δ is our control tolerance for the dial; it tells us how close our dial setting must be to the setting $x = c$ in order to guarantee that the diameter $f(x)$ of the shaft will be accurate to within ε of L . As the tolerance for error becomes stricter, we may have to adjust δ . The value of δ , how tight our control setting must be, depends on the value of ε , the error tolerance.

The definition of limit extends to functions on more general domains. It is only required that each open interval around c contain points in the domain of the function other than c . See Additional and Advanced Exercises 49–53 for examples of limits for functions with complicated domains. In the next section we will see how the definition of limit applies at points lying on the boundary of an interval.

Examples: Testing the Definition

The formal definition of limit does not tell how to find the limit of a function, but it does enable us to verify that a conjectured limit value is correct. The following examples show how the definition can be used to verify limit statements for specific functions. However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems so that the calculation of specific limits can be simplified, such as the theorems stated in the previous section.

**EXAMPLE 2** Show that

$$\lim_{x \rightarrow 1} (5x - 3) = 2.$$

Solution Set $c = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\varepsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $c = 1$, then it is true that $f(x)$ is within distance ε of $L = 2$. That is, we must show that if x satisfies

$$0 < |x - 1| < \delta,$$

then $f(x)$ will satisfy

$$|f(x) - 2| < \varepsilon.$$

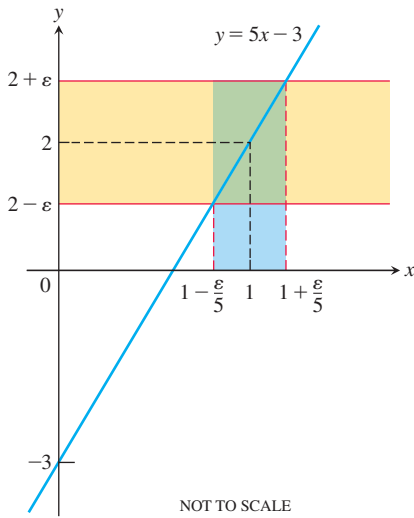


FIGURE 2.19 If $f(x) = 5x - 3$, then $0 < |x - 1| < \varepsilon/5$ guarantees that $|f(x) - 2| < \varepsilon$ (Example 2).

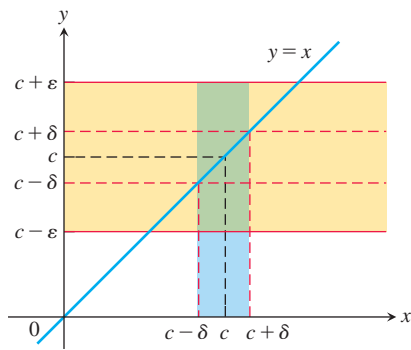


FIGURE 2.20 For the function $f(x) = x$, we find that $0 < |x - c| < \delta$ will guarantee $|f(x) - c| < \varepsilon$ whenever $\delta \leq \varepsilon$ (Example 3a).

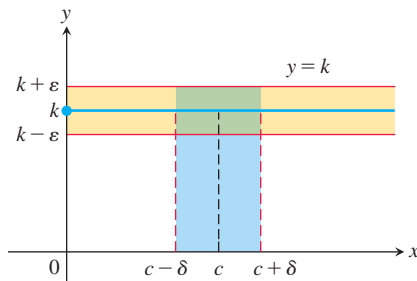


FIGURE 2.21 For the function $f(x) = k$, we find that $|f(x) - k| < \varepsilon$ for any positive δ (Example 3b).

We find δ by working backward from the ε -inequality, $|f(x) - 2| < \varepsilon$:

$$|(5x - 3) - 2| = |5x - 5| < \varepsilon \quad \text{Substitute.}$$

$$5|x - 1| < \varepsilon \quad \text{Factor.}$$

$$|x - 1| < \varepsilon/5. \quad \text{Simplify.}$$

Thus, we can take $\delta = \varepsilon/5$ (Figure 2.19). If $0 < |x - 1| < \delta = \varepsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < 5(\varepsilon/5) = \varepsilon,$$

which proves that $\lim_{x \rightarrow 1} (5x - 3) = 2$.

The value of $\delta = \varepsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|5x - 5| < \varepsilon$. Any smaller positive δ will do as well. The definition does not ask for the “best” positive δ , just one that will work. ■

EXAMPLE 3 Prove the following results, which were presented graphically in Section 2.2.

(a) $\lim_{x \rightarrow c} x = c$

(b) $\lim_{x \rightarrow c} k = k$ k constant

Solution

(a) Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$|x - c| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

The implication will hold if δ equals ε or any smaller positive number (Figure 2.20).

This proves that $\lim_{x \rightarrow c} x = c$.

(b) Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$|k - k| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Since $k - k = 0$, we will always have $|k - k| < \varepsilon$. Therefore we can use any positive number for δ , and the implication will hold (Figure 2.21). This proves that $\lim_{x \rightarrow c} k = k$. ■

Finding Deltas Algebraically for Given Epsilons

In Examples 2 and 3, the interval of values about c for which $|f(x) - L|$ was less than ε was symmetric about c and we could take δ to be half the length of that interval. When the interval around c on which we have $|f(x) - L| < \varepsilon$ is not symmetric about c , we can take δ to be the distance from c to the interval’s nearer endpoint.

EXAMPLE 4 For the limit $\lim_{x \rightarrow 5} \sqrt{x - 1} = 2$, find a $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that

$$|\sqrt{x - 1} - 2| < 1 \quad \text{whenever} \quad 0 < |x - 5| < \delta$$

Solution We organize the search into two steps.

1. Solve the inequality $|\sqrt{x - 1} - 2| < 1$ to find an interval containing $x = 5$ on which the inequality holds for all $x \neq 5$.

$$|\sqrt{x - 1} - 2| < 1$$

$$-1 < \sqrt{x - 1} - 2 < 1$$

$$1 < \sqrt{x - 1} < 3$$

$$1 < x - 1 < 9$$

$$2 < x < 10$$



FIGURE 2.22 An open interval of radius 3 about $x = 5$ will lie inside the open interval $(2, 10)$.

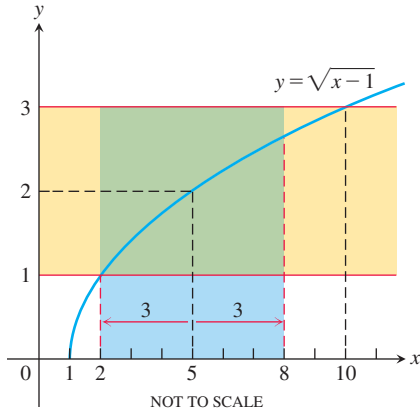


FIGURE 2.23 The function and intervals in Example 4.

The inequality holds for all x in the open interval $(2, 10)$, so it holds for all $x \neq 5$ in this interval as well.

- Find a value of $\delta > 0$ to place the centered interval $5 - \delta < x < 5 + \delta$ (centered at $x = 5$ inside the interval $(2, 10)$). The distance from 5 to the nearer endpoint of $(2, 10)$ is 3 (Figure 2.22). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < \delta$ will automatically place x between 2 and 10 and imply that $|\sqrt{x-1} - 2| < 1$ (Figure 2.23):

$$|\sqrt{x-1} - 2| < 1 \quad \text{whenever} \quad 0 < |x - 5| < 3.$$

How to Find Algebraically a δ for a Given f, L, c , and $\varepsilon > 0$

The process of finding a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta$$

can be accomplished in two steps.

- Solve the inequality $|f(x) - L| < \varepsilon$ to find an open interval (a, b) , containing c on which the inequality holds for all $x \neq c$. Note that we do not require the inequality to hold at $x = c$. It may hold there or it may not, but the value of f at $x = c$ does not influence the existence of a limit.
- Find a value of $\delta > 0$ that places the open interval $(c - \delta, c + \delta)$ centered at c inside the interval (a, b) . The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq c$ in this δ -interval.

EXAMPLE 5 Prove that $\lim_{x \rightarrow 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$$

Solution Our task is to show that given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - 4| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

- Solve the inequality $|f(x) - 4| < \varepsilon$ to find an open interval containing $x = 2$ on which the inequality holds for all $x \neq 2$.

For $x \neq c = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \varepsilon$:

$$\begin{aligned} |x^2 - 4| &< \varepsilon \\ -\varepsilon &< x^2 - 4 < \varepsilon \\ 4 - \varepsilon &< x^2 < 4 + \varepsilon \\ \sqrt{4 - \varepsilon} &< |x| < \sqrt{4 + \varepsilon} \quad \text{Assumes } \varepsilon < 4; \text{ see below.} \\ \sqrt{4 - \varepsilon} &< x < \sqrt{4 + \varepsilon}. \quad \text{An open interval about } x = 2 \\ &\quad \text{that solves the inequality} \end{aligned}$$

The inequality $|f(x) - 4| < \varepsilon$ holds for all $x \neq 2$ in the open interval $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$ (Figure 2.24).

- Find a value of $\delta > 0$ that places the centered interval $(2 - \delta, 2 + \delta)$ inside the interval $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$.

Take δ to be the distance from $x = 2$ to the nearer endpoint of $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$. In other words, take $\delta = \min\{2 - \sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2\}$, the *minimum* (the smaller) of the two numbers $2 - \sqrt{4 - \varepsilon}$ and $\sqrt{4 + \varepsilon} - 2$. If δ has this or any

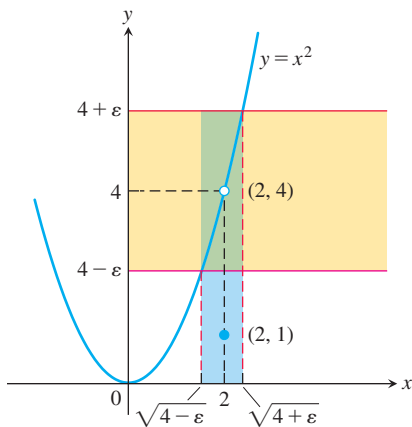


FIGURE 2.24 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \varepsilon$.

smaller positive value, the inequality $0 < |x - 2| < \delta$ will automatically place x between $\sqrt{4 - \varepsilon}$ and $\sqrt{4 + \varepsilon}$ to make $|f(x) - 4| < \varepsilon$. For all x ,

$$|f(x) - 4| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

This completes the proof for $\varepsilon < 4$.

If $\varepsilon \geq 4$, then we take δ to be the distance from $x = 2$ to the nearer endpoint of the interval $(0, \sqrt{4 + \varepsilon})$. In other words, take $\delta = \min\{2, \sqrt{4 + \varepsilon} - 2\}$. (See Figure 2.24.) ■

Using the Definition to Prove Theorems

We do not usually rely on the formal definition of limit to verify specific limits such as those in the preceding examples. Rather, we appeal to general theorems about limits, in particular the theorems of Section 2.2. The definition is used to prove these theorems (Appendix A.6). As an example, we prove part 1 of Theorem 1, the Sum Rule.

EXAMPLE 6 Given that $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$, prove that

$$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M.$$

Solution Let $\varepsilon > 0$ be given. We want to find a positive number δ such that

$$|f(x) + g(x) - (L + M)| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Regrouping terms, we get

$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| && \text{Triangle Inequality: } |a+b| \leq |a| + |b| \\ &\leq |f(x) - L| + |g(x) - M|. \end{aligned}$$

Since $\lim_{x \rightarrow c} f(x) = L$, there exists a number $\delta_1 > 0$ such that

$$|f(x) - L| < \varepsilon/2 \quad \text{whenever} \quad 0 < |x - c| < \delta_1. \quad \text{Can find } \delta_1 \text{ since } \lim_{x \rightarrow c} f(x) = L$$

Similarly, since $\lim_{x \rightarrow c} g(x) = M$, there exists a number $\delta_2 > 0$ such that

$$|g(x) - M| < \varepsilon/2 \quad \text{whenever} \quad 0 < |x - c| < \delta_2. \quad \text{Can find } \delta_2 \text{ since } \lim_{x \rightarrow c} g(x) = M$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2 . If $0 < |x - c| < \delta$ then $|x - c| < \delta_1$, so $|f(x) - L| < \varepsilon/2$, and $|x - c| < \delta_2$, so $|g(x) - M| < \varepsilon/2$. Therefore,

$$|f(x) + g(x) - (L + M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$. ■

EXERCISES 2.3

Centering Intervals About a Point

In Exercises 1–6, sketch the interval (a, b) , on the x -axis with the point c inside. Then find a value of $\delta > 0$ such that $a < x < b$ whenever $0 < |x - c| < \delta$.

1. $a = 1, \quad b = 7, \quad c = 5$

2. $a = 1, \quad b = 7, \quad c = 2$

3. $a = -7/2, \quad b = -1/2, \quad c = -3$

4. $a = -7/2, \quad b = -1/2, \quad c = -3/2$

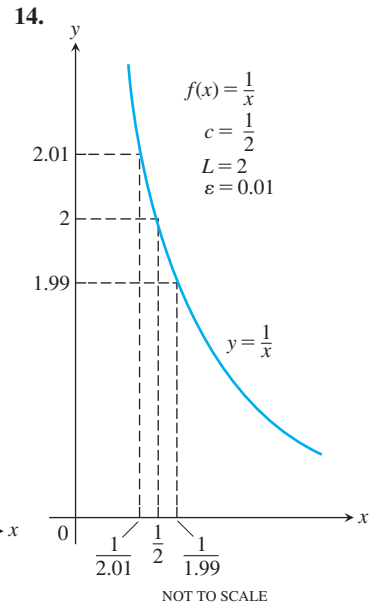
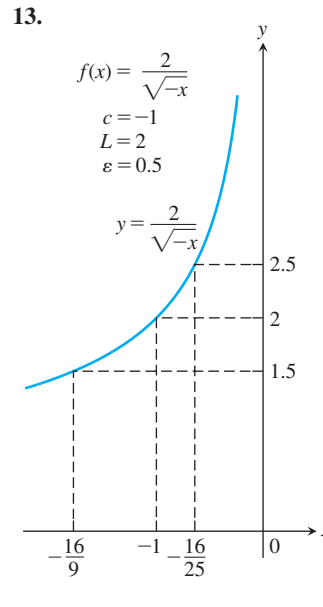
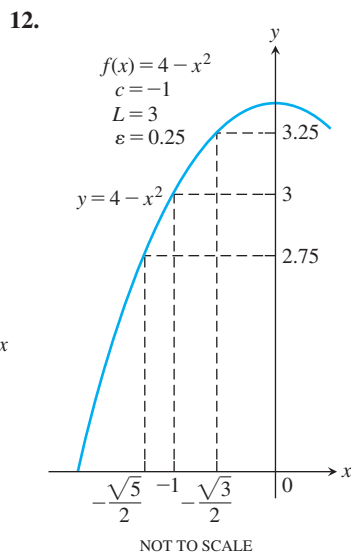
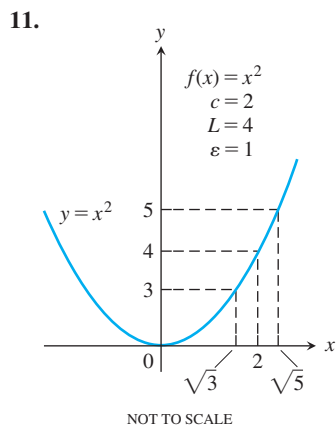
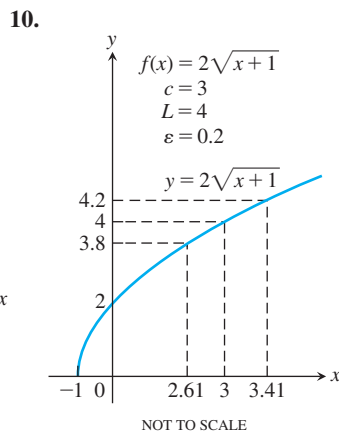
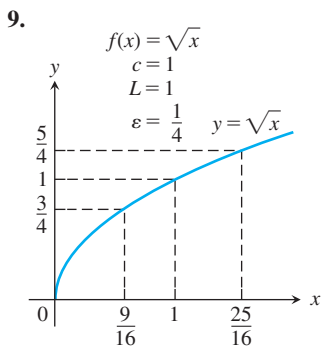
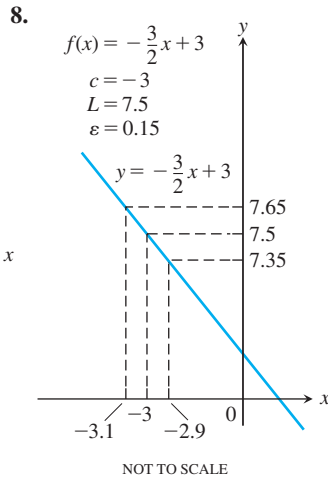
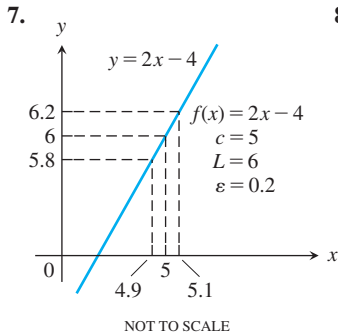
5. $a = 4/9, \quad b = 4/7, \quad c = 1/2$

6. $a = 2.7591, \quad b = 3.2391, \quad c = 3$

Finding Deltas Graphically

In Exercises 7–14, use the graphs to find a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$



Finding Deltas Algebraically

Each of Exercises 15–30 gives a function $f(x)$ and numbers L , c , and $\varepsilon > 0$. In each case, find the largest open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, the inequality $|f(x) - L| < \varepsilon$ holds.

15. $f(x) = x + 1$, $L = 5$, $c = 4$, $\varepsilon = 0.01$
16. $f(x) = 2x - 2$, $L = -6$, $c = -2$, $\varepsilon = 0.02$
17. $f(x) = \sqrt{x+1}$, $L = 1$, $c = 0$, $\varepsilon = 0.1$
18. $f(x) = \sqrt{x}$, $L = 1/2$, $c = 1/4$, $\varepsilon = 0.1$
19. $f(x) = \sqrt{19-x}$, $L = 3$, $c = 10$, $\varepsilon = 1$
20. $f(x) = \sqrt{x-7}$, $L = 4$, $c = 23$, $\varepsilon = 1$
21. $f(x) = 1/x$, $L = 1/4$, $c = 4$, $\varepsilon = 0.05$
22. $f(x) = x^2$, $L = 3$, $c = \sqrt{3}$, $\varepsilon = 0.1$
23. $f(x) = x^2$, $L = 4$, $c = -2$, $\varepsilon = 0.5$
24. $f(x) = 1/x$, $L = -1$, $c = -1$, $\varepsilon = 0.1$
25. $f(x) = x^2 - 5$, $L = 11$, $c = 4$, $\varepsilon = 1$
26. $f(x) = 120/x$, $L = 5$, $c = 24$, $\varepsilon = 1$
27. $f(x) = mx$, $m > 0$, $L = 2m$, $c = 2$, $\varepsilon = 0.03$
28. $f(x) = mx$, $m > 0$, $L = 3m$, $c = 3$, $\varepsilon = c > 0$
29. $f(x) = mx + b$, $m > 0$, $L = (m/2) + b$, $c = 1/2$, $\varepsilon = c > 0$
30. $f(x) = mx + b$, $m > 0$, $L = m + b$, $c = 1$, $\varepsilon = 0.05$

Using the Formal Definition

Each of Exercises 31–36 gives a function $f(x)$, a point c , and a positive number ε . Find $L = \lim_{x \rightarrow c} f(x)$. Then find a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

31. $f(x) = 3 - 2x$, $c = 3$, $\varepsilon = 0.02$
32. $f(x) = -3x - 2$, $c = -1$, $\varepsilon = 0.03$
33. $f(x) = \frac{x^2 - 4}{x - 2}$, $c = 2$, $\varepsilon = 0.05$

$$34. f(x) = \frac{x^2 + 6x + 5}{x + 5}, \quad c = -5, \quad \varepsilon = 0.05$$

$$35. f(x) = \sqrt{1 - 5x}, \quad c = -3, \quad \varepsilon = 0.5$$

$$36. f(x) = 4/x, \quad c = 2, \quad \varepsilon = 0.4$$

Prove the limit statements in Exercises 37–50.

$$37. \lim_{x \rightarrow 4} (9 - x) = 5 \qquad 38. \lim_{x \rightarrow 3} (3x - 7) = 2$$

$$39. \lim_{x \rightarrow 9} \sqrt{x - 5} = 2 \qquad 40. \lim_{x \rightarrow 0} \sqrt{4 - x} = 2$$

$$41. \lim_{x \rightarrow 1} f(x) = 1 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

$$42. \lim_{x \rightarrow -2} f(x) = 4 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq -2 \\ 1, & x = -2 \end{cases}$$

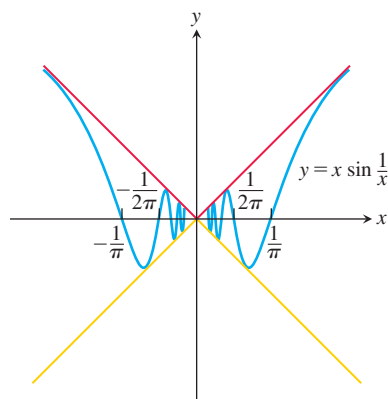
$$43. \lim_{x \rightarrow 1} \frac{1}{x} = 1 \qquad 44. \lim_{x \rightarrow \sqrt{3}} \frac{1}{x^2} = \frac{1}{3}$$

$$45. \lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = -6 \qquad 46. \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

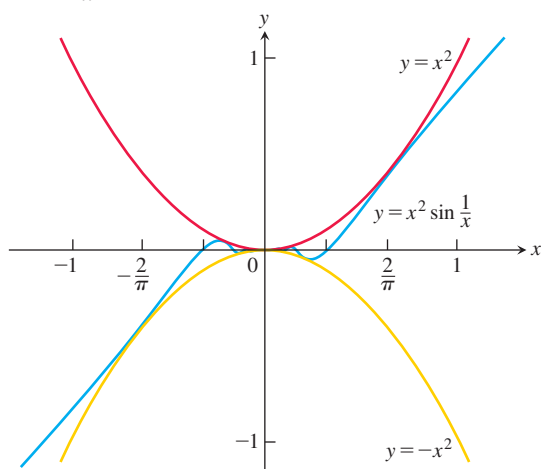
$$47. \lim_{x \rightarrow 1} f(x) = 2 \quad \text{if} \quad f(x) = \begin{cases} 4 - 2x, & x < 1 \\ 6x - 4, & x \geq 1 \end{cases}$$

$$48. \lim_{x \rightarrow 0} f(x) = 0 \quad \text{if} \quad f(x) = \begin{cases} 2x, & x < 0 \\ x/2, & x \geq 0 \end{cases}$$

$$49. \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$



$$50. \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$



Theory and Examples

51. Define what it means to say that $\lim_{x \rightarrow 0} g(x) = k$.

52. Prove that $\lim_{x \rightarrow c} f(x) = L$ if and only if $\lim_{h \rightarrow 0} f(h + c) = L$.

53. **A wrong statement about limits** Show by example that the following statement is wrong.

The number L is the limit of $f(x)$ as x approaches c if $f(x)$ gets closer to L as x approaches c .

Explain why the function in your example does not have the given value of L as a limit as $x \rightarrow c$.

54. **Another wrong statement about limits** Show by example that the following statement is wrong.

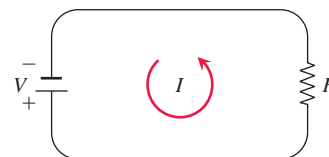
The number L is the limit of $f(x)$ as x approaches c if, given any $\varepsilon > 0$, there exists a value of x for which $|f(x) - L| < \varepsilon$.

Explain why the function in your example does not have the given value of L as a limit as $x \rightarrow c$.

T 55. Grinding engine cylinders Before contracting to grind engine cylinders to a cross-sectional area of 60 cm^2 , you need to know how much deviation from the ideal cylinder diameter of $c = 8.7404 \text{ cm}$ you can allow and still have the area come within 0.1 cm^2 of the required 60 cm^2 . To find out, you let $A = \pi(x/2)^2$ and look for the interval in which you must hold x to make $|A - 60| \leq 0.1$. What interval do you find?

56. Manufacturing electrical resistors

Ohm's law for electrical circuits like the one shown in the accompanying figure states that $V = RI$. In this equation, V is a constant voltage, I is the current in amperes, and R is the resistance in ohms. Your firm has been asked to supply the resistors for a circuit in which V will be 120 volts and I is to be 5 ± 0.1 amp. In what interval does R have to lie for I to be within 0.1 amp of the value $I_0 = 5$?



When Is a Number L Not the Limit of $f(x)$ As $x \rightarrow c$?

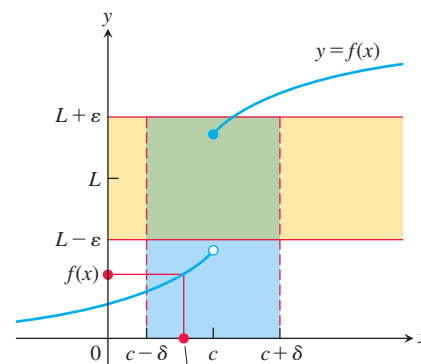
In Exercises 57–60 we will consider what it means to *not* have a limit.

Showing L is not a limit We can prove that $\lim_{x \rightarrow c} f(x) \neq L$ by providing an $\varepsilon > 0$ such that no possible $\delta > 0$ satisfies the condition

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

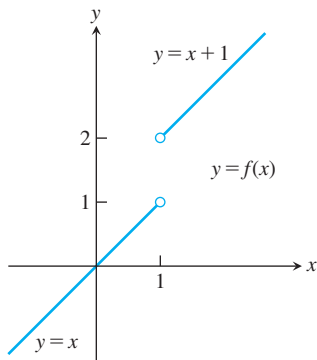
We accomplish this for our candidate ε by showing that for each $\delta > 0$ there exists a value of x such that

$$0 < |x - c| < \delta \quad \text{and} \quad |f(x) - L| \geq \varepsilon.$$



A value of x for which
 $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$

57. Let $f(x) = \begin{cases} x, & x < 1 \\ x + 1, & x > 1. \end{cases}$



- a. Let $\varepsilon = 1/2$. Show that no possible $\delta > 0$ satisfies the following condition:

$$|f(x) - 2| < 1/2 \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

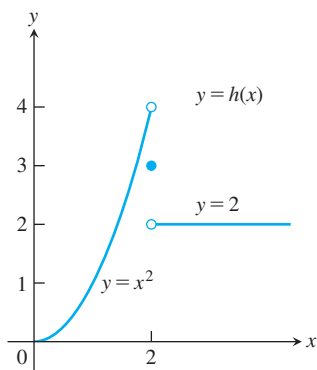
That is, show that for each $\delta > 0$, there is a value of x such that

$$0 < |x - 1| < \delta \quad \text{and} \quad |f(x) - 2| \geq 1/2.$$

This will show that $\lim_{x \rightarrow 1} f(x) \neq 2$.

- b. Show that $\lim_{x \rightarrow 1} f(x) \neq 1$.
c. Show that $\lim_{x \rightarrow 1} f(x) \neq 1.5$.

58. Let $h(x) = \begin{cases} x^2, & x < 2 \\ 3, & x = 2 \\ 2, & x > 2. \end{cases}$

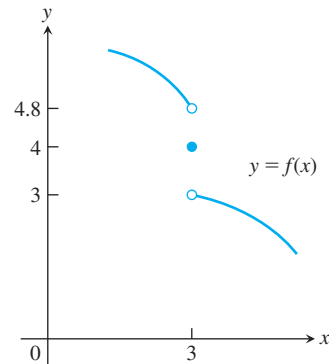


Show that

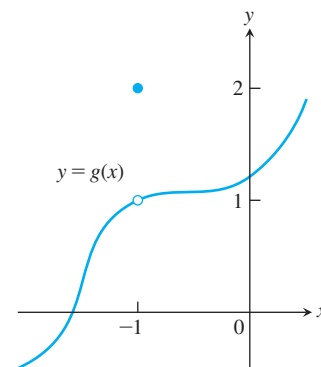
- a. $\lim_{x \rightarrow 2} h(x) \neq 4$
b. $\lim_{x \rightarrow 2} h(x) \neq 3$
c. $\lim_{x \rightarrow 2} h(x) \neq 2$

59. For the function graphed here, explain why

- a. $\lim_{x \rightarrow 3} f(x) \neq 4$
b. $\lim_{x \rightarrow 3} f(x) \neq 4.8$
c. $\lim_{x \rightarrow 3} f(x) \neq 3$



60. a. For the function graphed here, show that $\lim_{x \rightarrow -1} g(x) \neq 2$.
b. Does $\lim_{x \rightarrow -1} g(x)$ appear to exist? If so, what is the value of the limit? If not, why not?



COMPUTER EXPLORATIONS

In Exercises 61–66, you will further explore finding deltas graphically. Use a CAS to perform the following steps:

- Plot the function $y = f(x)$ near the point c being approached.
- Guess the value of the limit L and then evaluate the limit symbolically to see if you guessed correctly.
- Using the value $\varepsilon = 0.2$, graph the banding lines $y_1 = L - \varepsilon$ and $y_2 = L + \varepsilon$ together with the function f near c .
- From your graph in part (c), estimate a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Test your estimate by plotting f , y_1 , and y_2 over the interval $0 < |x - c| < \delta$. For your viewing window use $c - 2\delta \leq x \leq c + 2\delta$ and $L - 2\varepsilon \leq y \leq L + 2\varepsilon$. If any function values lie outside the interval $[L - \varepsilon, L + \varepsilon]$, your choice of δ was too large. Try again with a smaller estimate.

- Repeat parts (c) and (d) successively for $\varepsilon = 0.1, 0.05$, and 0.001 .

61. $f(x) = \frac{x^4 - 81}{x - 3}$, $c = 3$ 62. $f(x) = \frac{5x^3 + 9x^2}{2x^5 + 3x^2}$, $c = 0$
63. $f(x) = \frac{\sin 2x}{3x}$, $c = 0$ 64. $f(x) = \frac{x(1 - \cos x)}{x - \sin x}$, $c = 0$
65. $f(x) = \frac{\sqrt[3]{x} - 1}{x - 1}$, $c = 1$
66. $f(x) = \frac{3x^2 - (7x + 1)\sqrt{x} + 5}{x - 1}$, $c = 1$

2.4 One-Sided Limits

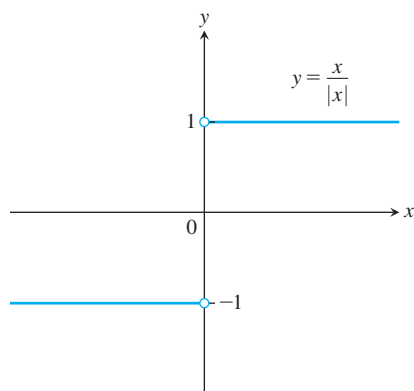


FIGURE 2.25 Different right-hand and left-hand limits at the origin.

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side (where $x > c$) only. These allow us to describe functions that have different limits at a point, depending on whether we approach the point from the left or from the right. One-sided limits also allow us to say what it means for a function to have a limit at an endpoint of an interval.

Approaching a Limit from One Side

Suppose a function f is defined on an interval that extends to both sides of a number c . In order for f to have a limit L as x approaches c , the values of $f(x)$ must approach the value L as x approaches c from either side. Because of this, we sometimes say that the limit is **two-sided**.

If f fails to have a two-sided limit at c , it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit** or **limit from the right**. Similarly, a **left-hand limit** is also called a **limit from the left**.

The function $f(x) = x/|x|$ (Figure 2.25) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if we consider only the values of $f(x)$ on an interval (c, b) , where $c < b$, and the values of $f(x)$ become arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c . In this case we write

$$\lim_{x \rightarrow c^+} f(x) = L.$$

The notation “ $x \rightarrow c^+$ ” means that we consider only values of $f(x)$ for x greater than c . We don’t consider values of $f(x)$ for $x \leq c$.

Similarly, if $f(x)$ is defined on an interval (a, c) , where $a < c$, and $f(x)$ approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c . We write

$$\lim_{x \rightarrow c^-} f(x) = M.$$

The symbol “ $x \rightarrow c^-$ ” means that we consider the values of f only at x -values less than c .

These informal definitions of one-sided limits are illustrated in Figure 2.26. For the function $f(x) = x/|x|$ in Figure 2.25 we have

$$\lim_{x \rightarrow 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} f(x) = -1.$$

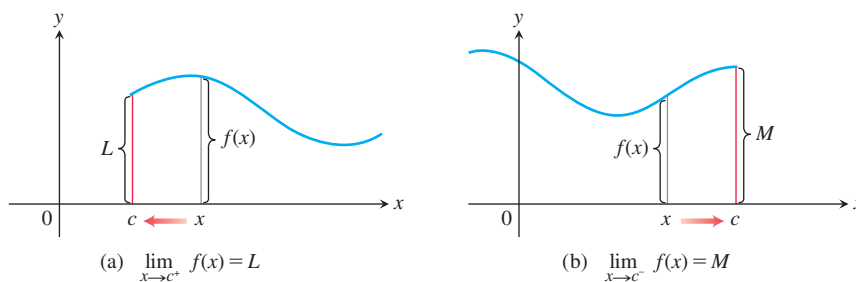


FIGURE 2.26 (a) Right-hand limit as x approaches c . (b) Left-hand limit as x approaches c .

One-sided limits have all the properties listed in Theorem 1 in Section 2.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as does the Sandwich Theorem. One-sided limits are related to limits at interior points in the following way.

THEOREM 5

Suppose that a function f is defined on an open interval containing c , except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has both a limit from the left at c and a limit from the right at c , and these one-sided limits are equal:

$$\lim_{x \rightarrow c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$

Theorem 5 applies at interior points of a function's domain. At a boundary point of an interval in its domain, a function has a limit when it has an appropriate one-sided limit.

Limits at Endpoints of an Interval

- If f is defined on an open interval (b, c) to the left of c and not defined on an open interval (c, d) to the right of c , then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c^-} f(x).$$

- If f is defined on an open interval (c, d) to the right of c and not defined on an open interval (b, c) to the left of c , then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c^+} f(x).$$

(The definition of a limit on an arbitrary domain is discussed in Additional and Advanced Exercises 39–42.)

EXAMPLE 1 For the function graphed in Figure 2.27,

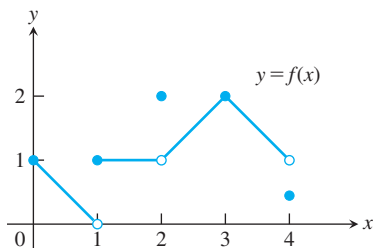


FIGURE 2.27 Graph of the function in Example 1.

- At $x = 0$: $\lim_{x \rightarrow 0^-} f(x)$ does not exist, f is not defined to the left of $x = 0$.
 $\lim_{x \rightarrow 0^+} f(x) = 1$, f has a right-hand limit at $x = 0$.
 $\lim_{x \rightarrow 0} f(x) = 1$. f has a limit at domain endpoint $x = 0$.
- At $x = 1$: $\lim_{x \rightarrow 1^-} f(x) = 0$, Even though $f(1) = 1$.
 $\lim_{x \rightarrow 1^+} f(x) = 1$,
 $\lim_{x \rightarrow 1} f(x)$ does not exist. Right- and left-hand limits are not equal.
- At $x = 2$: $\lim_{x \rightarrow 2^-} f(x) = 1$,
 $\lim_{x \rightarrow 2^+} f(x) = 1$,
 $\lim_{x \rightarrow 2} f(x) = 1$. Even though $f(2) = 2$.
- At $x = 3$: $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3} f(x) = f(3) = 2$.
- At $x = 4$: $\lim_{x \rightarrow 4^-} f(x) = 1$, Even though $f(4) \neq 1$.
 $\lim_{x \rightarrow 4^+} f(x)$ does not exist, f is not defined to the right of $x = 4$.
 $\lim_{x \rightarrow 4} f(x) = 1$. f has a limit at domain endpoint $x = 4$.

At every other point c in $[0, 4]$, $f(x)$, has limit $f(c)$. ■

EXAMPLE 2 The domain of the function $\arcsin x$ is a union of the intervals $(-\infty, -1]$ and $[1, \infty)$, as shown in Figure 2.28. For the boundary points of these intervals,

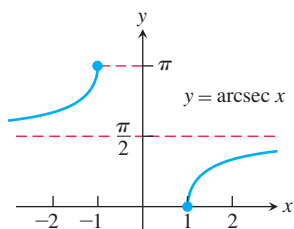


FIGURE 2.28 The arcsin function has limits at $x = \pm 1$.

- At $x = -1$: $\lim_{x \rightarrow -1^-} \arcsin x = \pi$, $\arcsin x$ has a limit from the left at $x = -1$.
 $\lim_{x \rightarrow -1^+} \arcsin x$ does not exist, $\arcsin x$ is not defined on $(-1, 1)$.
 $\lim_{x \rightarrow -1} \arcsin x = \pi$. $\arcsin x$ has a limit at $x = -1$.

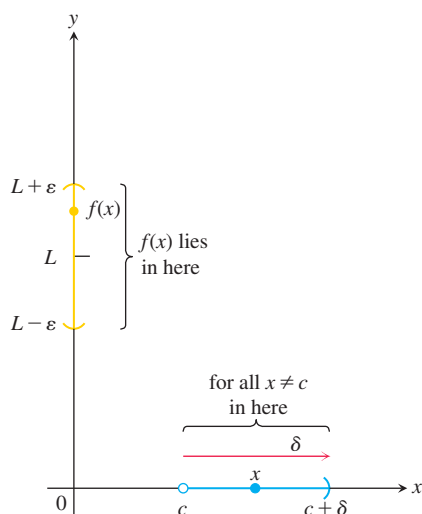


FIGURE 2.29 Intervals associated with the definition of right-hand limit.

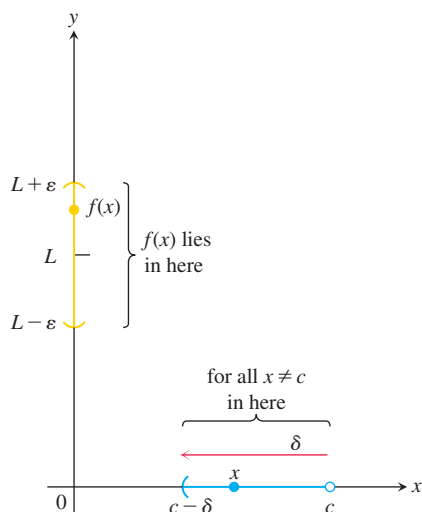


FIGURE 2.30 Intervals associated with the definition of left-hand limit.

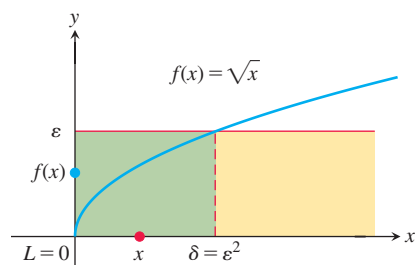


FIGURE 2.31 $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$ in Example 3.

At $x = 1$: $\lim_{x \rightarrow 1^-} \operatorname{arcsec} x$ does not exist, $\operatorname{arcsec} x$ is not defined on $(-1, 1)$.
 $\lim_{x \rightarrow 1^+} \operatorname{arcsec} x = 0$, $\operatorname{arcsec} x$ has a limit from the right at $x = 1$.
 $\lim_{x \rightarrow 1} \operatorname{arcsec} x = 0$. $\operatorname{arcsec} x$ has a limit at $x = 1$.

At every c in $(-\infty, -1)$ and every c in $(1, \infty)$, the limit of $\operatorname{arcsec} x$ as $x \rightarrow c$ is equal to $\operatorname{arcsec} c$. However, for each c in the interval $(-1, 1)$, $\lim_{x \rightarrow c} \operatorname{arcsec} x$ does not exist. ■

Precise Definitions of One-Sided Limits

The formal definition of the limit in Section 2.3 is readily modified for one-sided limits.

DEFINITIONS (a) Assume the domain of f contains an interval (c, d) to the right of c . We say that $f(x)$ has **right-hand limit** L at c , and write

$$\lim_{x \rightarrow c^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta.$$

(b) Assume the domain of f contains an interval (b, c) to the left of c . We say that f has **left-hand limit** L at c , and write

$$\lim_{x \rightarrow c^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c - \delta < x < c.$$

The definitions are illustrated in Figures 2.29 and 2.30.

EXAMPLE 3 Prove that

$$\lim_{x \rightarrow 0^+} \sqrt{x} = 0.$$

Solution Let $\varepsilon > 0$ be given. Here $c = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that

$$|\sqrt{x} - 0| < \varepsilon \quad \text{whenever} \quad 0 < x < \delta,$$

or

$$\sqrt{x} < \varepsilon \quad \text{whenever} \quad 0 < x < \delta. \quad \sqrt{x} \geq 0 \text{ so } |\sqrt{x}| = \sqrt{x}$$

Squaring both sides of this last inequality gives

$$x < \varepsilon^2 \quad \text{if} \quad 0 < x < \delta.$$

If we choose $\delta = \varepsilon^2$ we have

$$\sqrt{x} < \varepsilon \quad \text{whenever} \quad 0 < x < \delta = \varepsilon^2,$$

or

$$|\sqrt{x} - 0| < \varepsilon \quad \text{whenever} \quad 0 < x < \varepsilon^2.$$

According to the definition, this shows that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$ (Figure 2.31).

Note that since 0 is an endpoint of the domain where $\sqrt{x}$ is defined, it is also true that $\lim_{x \rightarrow 0} \sqrt{x} = 0$. ■

The functions examined so far have had some kind of limit at each point of interest. In general, that need not be the case.

EXAMPLE 4 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Figure 2.32).

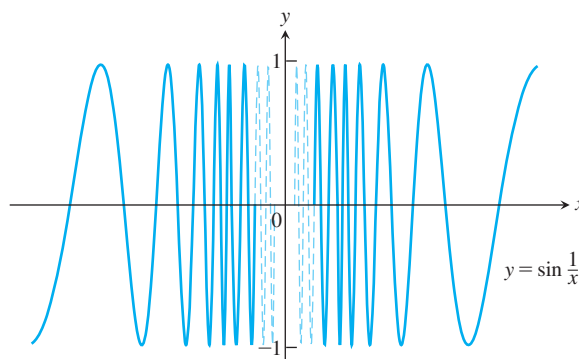


FIGURE 2.32 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y -axis.

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound, and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1 . There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ■

Limits Involving $(\sin \theta)/\theta$

A central fact about $(\sin \theta)/\theta$ is that in radian measure its limit as $\theta \rightarrow 0$ is 1. We can see this in Figure 2.33 and confirm it algebraically using the Sandwich Theorem. You will see the importance of this limit in Section 3.5, where instantaneous rates of change of the trigonometric functions are studied.

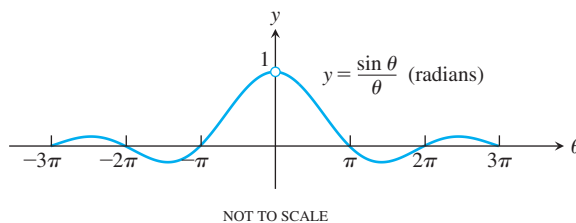


FIGURE 2.33 The graph of $f(\theta) = (\sin \theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

THEOREM 6—Limit of the Ratio $\sin \theta/\theta$ as $\theta \rightarrow 0$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \text{ in radians}) \quad (1)$$

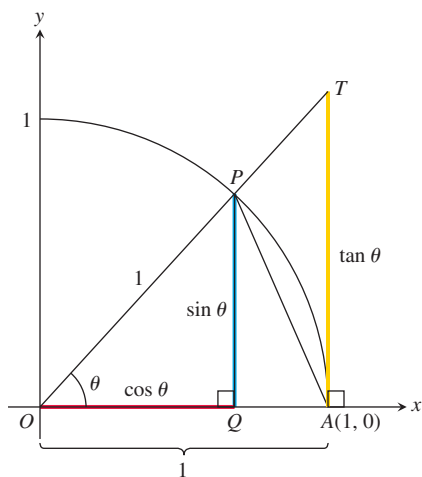


FIGURE 2.34 The ratio $TA/OA = \tan \theta$, and $OA = 1$, so $TA = \tan \theta$.

The use of radians to measure angles is essential in Equation (2): The area of sector OAP is $\theta/2$ only if θ is measured in radians.

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of θ less than $\pi/2$ (Figure 2.34). Notice that

$$\text{Area } \triangle OAP < \text{area sector } OAP < \text{area } \triangle OAT.$$

We can express these areas in terms of θ as follows:

$$\begin{aligned} \text{Area } \triangle OAP &= \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2}(1)(\sin \theta) = \frac{1}{2} \sin \theta \\ \text{Area sector } OAP &= \frac{1}{2} r^2 \theta = \frac{1}{2}(1)^2 \theta = \frac{\theta}{2} \\ \text{Area } \triangle OAT &= \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2}(1)(\tan \theta) = \frac{1}{2} \tan \theta. \end{aligned} \quad (2)$$

$$\frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2) \sin \theta$, which is positive, since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

Since $\lim_{\theta \rightarrow 0^+} \cos \theta = 1$ (Example 12b, Section 2.2), the Sandwich Theorem gives

$$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1.$$

To consider the left-hand limit, we recall that $\sin \theta$ and θ are both *odd functions* (Section 1.1). Therefore, $f(\theta) = (\sin \theta)/\theta$ is an *even function*, with a graph symmetric about the y-axis (see Figure 2.33). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1 = \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta},$$

so $\lim_{\theta \rightarrow 0} (\sin \theta)/\theta = 1$ by Theorem 5. ■

EXAMPLE 5 Show that (a) $\lim_{y \rightarrow 0} \frac{\cos y - 1}{y} = 0$ and (b) $\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos y = 1 - 2 \sin^2(y/2)$, we calculate

$$\begin{aligned} \lim_{y \rightarrow 0} \frac{\cos y - 1}{y} &= \lim_{y \rightarrow 0} \frac{2 \sin^2(y/2)}{y} \\ &= -\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \sin \theta && \text{Let } \theta = y/2. \\ &= -(1)(0) = 0. && \text{Eq. (1) and Example 12a in Section 2.2} \end{aligned}$$

- (b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} &= \lim_{x \rightarrow 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x} \\ &= \frac{2}{5} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} && \text{Eq. (1) applies with } \theta = 2x. \\ &= \frac{2}{5}(1) = \frac{2}{5}.\end{aligned}$$

EXAMPLE 6 Find $\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t}$.

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned}\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t} &= \lim_{t \rightarrow 0} \frac{1}{3} \cdot \frac{1}{t} \cdot \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t} \\ &= \frac{1}{3} \lim_{t \rightarrow 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t} \\ &= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.\end{aligned}$$

Eq. (1) and Example 12b in Section 2.2

EXAMPLE 7 Show that for nonzero constants A and B ,

$$\lim_{\theta \rightarrow 0} \frac{\sin A\theta}{\sin B\theta} = \frac{A}{B}.$$

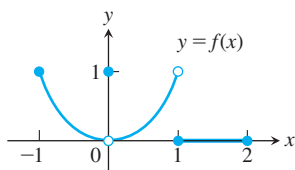
Solution

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{\sin A\theta}{\sin B\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin A\theta}{A\theta} \cdot A\theta \cdot \frac{B\theta}{\sin B\theta} \cdot \frac{1}{B\theta} && \text{Multiply and divide by } A\theta \text{ and } B\theta. \\ &= \lim_{\theta \rightarrow 0} \frac{\sin A\theta}{A\theta} \cdot \frac{B\theta}{\sin B\theta} \cdot \frac{A}{B} && \lim_{u \rightarrow 0} \frac{\sin u}{u} = 1, \text{ with } u = A\theta \\ &= \lim_{\theta \rightarrow 0} (1)(1) \cdot \frac{A}{B} && \lim_{v \rightarrow 0} \frac{v}{\sin v} = 1, \text{ with } v = B\theta \\ &= \frac{A}{B}.\end{aligned}$$

EXERCISES 2.4

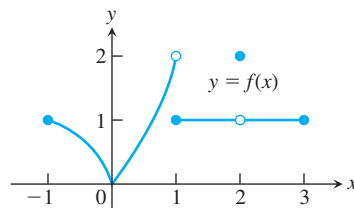
Finding Limits Graphically

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



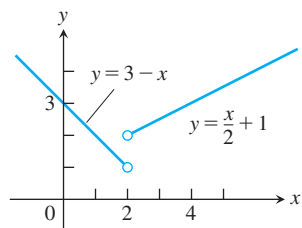
- | | |
|---|--|
| a. $\lim_{x \rightarrow -1^+} f(x) = 1$ | b. $\lim_{x \rightarrow 0^-} f(x) = 0$ |
| c. $\lim_{x \rightarrow 0^-} f(x) = 1$ | d. $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$ |
| e. $\lim_{x \rightarrow 0} f(x)$ exists. | f. $\lim_{x \rightarrow 0} f(x) = 0$ |
| g. $\lim_{x \rightarrow 0} f(x) = 1$ | h. $\lim_{x \rightarrow 1} f(x) = 1$ |
| i. $\lim_{x \rightarrow 1} f(x) = 0$ | j. $\lim_{x \rightarrow 2^-} f(x) = 2$ |
| k. $\lim_{x \rightarrow -1^-} f(x)$ does not exist. | l. $\lim_{x \rightarrow 2^+} f(x) = 0$ |

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



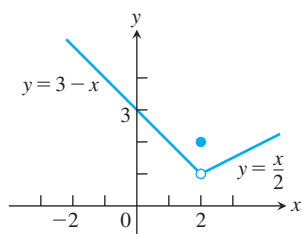
- | | |
|---|--|
| a. $\lim_{x \rightarrow -1^+} f(x) = 1$ | b. $\lim_{x \rightarrow 2} f(x)$ does not exist. |
| c. $\lim_{x \rightarrow 2} f(x) = 2$ | d. $\lim_{x \rightarrow 1^-} f(x) = 2$ |
| e. $\lim_{x \rightarrow 1^+} f(x) = 1$ | f. $\lim_{x \rightarrow 1} f(x)$ does not exist. |
| g. $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)$ | |
| h. $\lim_{x \rightarrow c} f(x)$ exists at every c in the open interval $(-1, 1)$. | |
| i. $\lim_{x \rightarrow c} f(x)$ exists at every c in the open interval $(1, 3)$. | |
| j. $\lim_{x \rightarrow -1^-} f(x) = 0$ | k. $\lim_{x \rightarrow 3^+} f(x)$ does not exist. |

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2. \end{cases}$



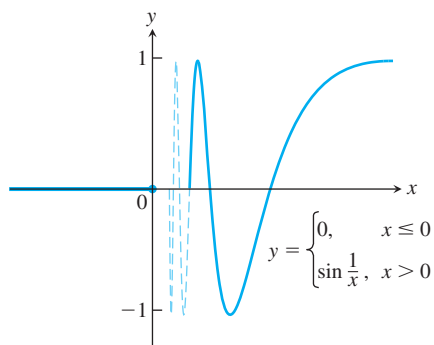
- Find $\lim_{x \rightarrow 2^-} f(x)$ and $\lim_{x \rightarrow 2^+} f(x)$.
- Does $\lim_{x \rightarrow 2} f(x)$ exist? If so, what is it? If not, why not?
- Find $\lim_{x \rightarrow 4^-} f(x)$ and $\lim_{x \rightarrow 4^+} f(x)$.
- Does $\lim_{x \rightarrow 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \frac{x}{2}, & x > 2. \end{cases}$



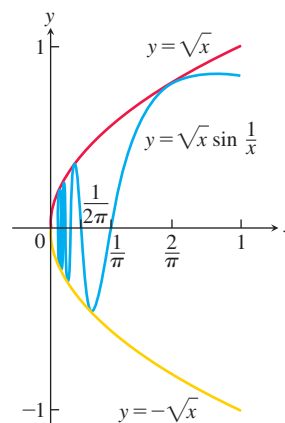
- Find $\lim_{x \rightarrow 2^-} f(x)$, $\lim_{x \rightarrow 2^+} f(x)$, and $f(2)$.
- Does $\lim_{x \rightarrow 2} f(x)$ exist? If so, what is it? If not, why not?
- Find $\lim_{x \rightarrow -1^-} f(x)$ and $\lim_{x \rightarrow -1^+} f(x)$.
- Does $\lim_{x \rightarrow -1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0. \end{cases}$



- Does $\lim_{x \rightarrow 0^+} f(x)$ exist? If so, what is it? If not, why not?
- Does $\lim_{x \rightarrow 0^-} f(x)$ exist? If so, what is it? If not, why not?
- Does $\lim_{x \rightarrow 0} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x} \sin(1/x)$.



- Does $\lim_{x \rightarrow 0^+} g(x)$ exist? If so, what is it? If not, why not?
- Does $\lim_{x \rightarrow 0^-} g(x)$ exist? If so, what is it? If not, why not?
- Does $\lim_{x \rightarrow 0} g(x)$ exist? If so, what is it? If not, why not?

7. a. Graph $f(x) = \begin{cases} x^3, & x \neq 1 \\ 0, & x = 1. \end{cases}$

- Find $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$.
- Does $\lim_{x \rightarrow 1} f(x)$ exist? If so, what is it? If not, why not?

8. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \neq 1 \\ 2, & x = 1. \end{cases}$

- Find $\lim_{x \rightarrow 1^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$.
- Does $\lim_{x \rightarrow 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

- What are the domain and range of f ?
- At what points c , if any, does $\lim_{x \rightarrow c} f(x)$ exist?
- At what points does the left-hand limit exist but not the right-hand limit?
- At what points does the right-hand limit exist but not the left-hand limit?

9. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 2, & x = 2 \end{cases}$

10. $f(x) = \begin{cases} x, & -1 \leq x < 0, \text{ or } 0 < x \leq 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$

Finding One-Sided Limits Algebraically

Find the limits in Exercises 11–20.

11. $\lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x+1}}$ 12. $\lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x+2}}$

13. $\lim_{x \rightarrow -2^+} \left(\frac{x}{x+1} \right) \left(\frac{2x+5}{x^2+x} \right)$

14. $\lim_{x \rightarrow 1^-} \left(\frac{1}{x+1} \right) \left(\frac{x+6}{x} \right) \left(\frac{3-x}{7} \right)$

15. $\lim_{h \rightarrow 0^+} \frac{\sqrt{h^2 + 4h + 5} - \sqrt{5}}{h}$

16. $\lim_{h \rightarrow 0^-} \frac{\sqrt{6} - \sqrt{5h^2 + 11h + 6}}{h}$

17. a. $\lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2}$ b. $\lim_{x \rightarrow -2^-} (x+3) \frac{|x+2|}{x+2}$

18. a. $\lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$ b. $\lim_{x \rightarrow 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|}$

19. a. $\lim_{x \rightarrow 0^+} \frac{|\sin x|}{x}$ b. $\lim_{x \rightarrow 0^-} \frac{|\sin x|}{x}$

20. a. $\lim_{x \rightarrow 0^+} \frac{1 - \cos x}{|\cos x - 1|}$ b. $\lim_{x \rightarrow 0^-} \frac{\cos x - 1}{|\cos x - 1|}$

Use the graph of the greatest integer function $y = \lfloor x \rfloor$, Figure 1.10 in Section 1.1, to help you find the limits in Exercises 21 and 22.

21. a. $\lim_{\theta \rightarrow 3^+} \frac{|\theta|}{\theta}$ b. $\lim_{\theta \rightarrow 3^-} \frac{|\theta|}{\theta}$

22. a. $\lim_{t \rightarrow 4^+} (t - \lfloor t \rfloor)$ b. $\lim_{t \rightarrow 4^-} (t - \lfloor t \rfloor)$

Using $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

Find the limits in Exercises 23–46.

23. $\lim_{\theta \rightarrow 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta}$

24. $\lim_{t \rightarrow 0} \frac{\sin kt}{t} \quad (k \text{ constant})$

25. $\lim_{y \rightarrow 0} \frac{\sin 3y}{4y}$

26. $\lim_{h \rightarrow 0^-} \frac{h}{\sin 3h}$

27. $\lim_{x \rightarrow 0} \frac{\tan 2x}{x}$

28. $\lim_{t \rightarrow 0} \frac{2t}{\tan t}$

29. $\lim_{x \rightarrow 0} \frac{x \csc 2x}{\cos 5x}$

30. $\lim_{x \rightarrow 0} 6x^2 (\cot x) (\csc 2x)$

31. $\lim_{x \rightarrow 0} \frac{x + x \cos x}{\sin x \cos x}$

32. $\lim_{x \rightarrow 0} \frac{x^2 - x + \sin x}{2x}$

33. $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin 2\theta}$

34. $\lim_{x \rightarrow 0} \frac{x - x \cos x}{\sin^2 3x}$

35. $\lim_{t \rightarrow 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$

36. $\lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h}$

37. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta}$

38. $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 4x}$

39. $\lim_{\theta \rightarrow 0} \theta \cos \theta$

40. $\lim_{\theta \rightarrow 0} \sin \theta \cot \theta$

41. $\lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 8x}$

42. $\lim_{y \rightarrow 0} \frac{\sin 3y \cot 5y}{y \cot 4y}$

43. $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta^2 \cot 3\theta}$

44. $\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

45. $\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{2x}$

46. $\lim_{x \rightarrow 0} \frac{\cos^2 x - \cos x}{x^2}$

Theory and Examples

47. Once you know $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow a^-} f(x)$ at an interior point of the domain of f , do you then know $\lim_{x \rightarrow a} f(x)$? Give reasons for your answer.

48. If you know that $\lim_{x \rightarrow c} f(x)$ exists at an interior point of a domain interval of f , can you find its value by calculating $\lim_{x \rightarrow c^+} f(x)$? Give reasons for your answer.

49. Suppose that f is an odd function of x . Does knowing that $\lim_{x \rightarrow 0^+} f(x) = 3$ tell you anything about $\lim_{x \rightarrow 0^-} f(x)$? Give reasons for your answer.

50. Suppose that f is an even function of x . Does knowing that $\lim_{x \rightarrow 2^-} f(x) = 7$ tell you anything about either $\lim_{x \rightarrow -2^-} f(x)$ or $\lim_{x \rightarrow -2^+} f(x)$? Give reasons for your answer.

Formal Definitions of One-Sided Limits

51. Given $\varepsilon > 0$, find an interval $I = (5, 5 + \delta)$, $\delta > 0$, such that if x lies in I , then $\sqrt{x} - 5 < \varepsilon$. What limit is being verified and what is its value?

52. Given $\varepsilon > 0$, find an interval $I = (4 - \delta, 4)$, $\delta > 0$, such that if x lies in I , then $\sqrt{4 - x} < \varepsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 53 and 54.

53. $\lim_{x \rightarrow 0^-} \frac{x}{|x|} = -1$

54. $\lim_{x \rightarrow 2^+} \frac{x-2}{|x-2|} = 1$

55. **Greatest integer function** Find (a) $\lim_{x \rightarrow 400^+} \lfloor x \rfloor$ and (b) $\lim_{x \rightarrow 400^-} \lfloor x \rfloor$; then use limit definitions to verify your findings. (c) Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x \rightarrow 400} \lfloor x \rfloor$? Give reasons for your answer.

56. **One-sided limits** Let $f(x) = \begin{cases} x^2 \sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0. \end{cases}$

Find (a) $\lim_{x \rightarrow 0^+} f(x)$ and (b) $\lim_{x \rightarrow 0^-} f(x)$; then use limit definitions to verify your findings. (c) Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x \rightarrow 0} f(x)$? Give reasons for your answer.

2.5 Limits Involving Infinity; Asymptotes of Graphs

In this section we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x \rightarrow \pm\infty$. We further extend the concept of limit to *infinite limits*. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude. We use these ideas to analyze the graphs of functions having *horizontal* or *vertical asymptotes*.

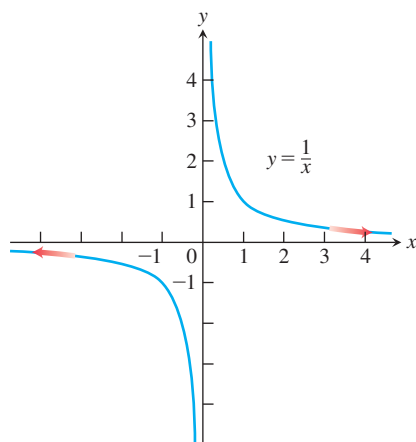


FIGURE 2.35 The graph of $y = 1/x$ approaches 0 as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

Finite Limits as $x \rightarrow \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, the function $f(x) = 1/x$ is defined for all $x \neq 0$ (Figure 2.35). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit 0 as $x \rightarrow \infty$ or $x \rightarrow -\infty$, or that 0 is a *limit of $f(x) = 1/x$ at infinity and at negative infinity*. Here are precise definitions for the limit of a function whose domain contains positive or negative numbers of unbounded magnitude.

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \rightarrow \infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M.$$

2. We say that $f(x)$ has the **limit L as x approaches negative infinity** and write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x < N.$$

Intuitively, $\lim_{x \rightarrow \infty} f(x) = L$ if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L . Similarly, $\lim_{x \rightarrow -\infty} f(x) = L$ if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L .

The strategy for calculating limits of functions as $x \rightarrow +\infty$ or as $x \rightarrow -\infty$ is similar to the one for finite limits in Section 2.2. There we first found the limits of the constant and identity functions $y = k$ and $y = x$. We then extended these results to other functions by applying Theorem 1 on limits of algebraic combinations. Here we do the same thing, except that the starting functions are $y = k$ and $y = 1/x$ instead of $y = k$ and $y = x$.

The basic facts to be verified by applying the formal definition are

$$\lim_{x \rightarrow \pm\infty} k = k \quad \text{and} \quad \lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0. \quad (1)$$

We prove the second result in Example 1, and leave the first to Exercises 93 and 94.

EXAMPLE 1 Show that

$$(a) \quad \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$(b) \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

Solution

- (a) Let $\varepsilon > 0$ be given. We must find a number M such that

$$\left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| < \varepsilon \quad \text{whenever} \quad x > M.$$

The implication will hold if $M = 1/\varepsilon$ or any larger positive number (Figure 2.36). This proves $\lim_{x \rightarrow \infty} (1/x) = 0$.

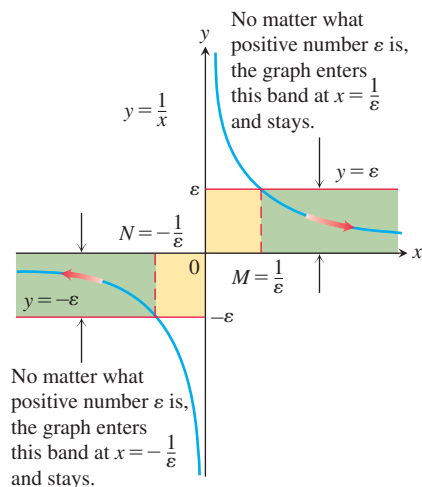


FIGURE 2.36 The geometry behind the argument in Example 1.

(b) Let $\varepsilon > 0$ be given. We must find a number N such that

$$\left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| < \varepsilon \quad \text{whenever} \quad x < N.$$

The implication will hold if $N = -1/\varepsilon$ or any number less than $-1/\varepsilon$ (Figure 2.36). This proves $\lim_{x \rightarrow -\infty} (1/x) = 0$. ■

Limits at infinity have properties similar to those of finite limits.

THEOREM 7

All the Limit Laws in Theorem 1 are true when we replace $\lim$ by $\lim_{x \rightarrow c}$, $\lim_{x \rightarrow \infty}$, or $\lim_{x \rightarrow -\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

EXAMPLE 2 The properties in Theorem 7 are used to calculate limits in the same way as when x approaches a finite number c .

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow \infty} \left(5 + \frac{1}{x} \right) &= \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{1}{x} && \text{Sum Rule} \\ &= 5 + 0 = 5 && \text{Known limits} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow -\infty} \frac{\pi\sqrt{3}}{x^2} &= \lim_{x \rightarrow -\infty} \pi\sqrt{3} \cdot \frac{1}{x} \cdot \frac{1}{x} \\ &= \lim_{x \rightarrow -\infty} \pi\sqrt{3} \cdot \lim_{x \rightarrow -\infty} \frac{1}{x} \cdot \lim_{x \rightarrow -\infty} \frac{1}{x} && \text{Product Rule} \\ &= \pi\sqrt{3} \cdot 0 \cdot 0 = 0 && \text{Known limits} \end{aligned}$$

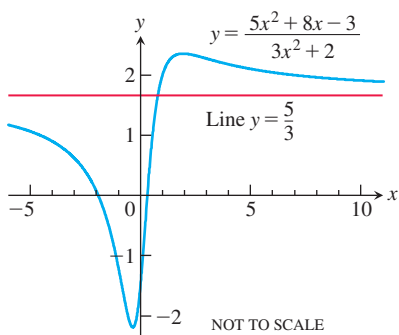


FIGURE 2.37 The graph of the function in Example 3a. The graph approaches the line $y = 5/3$ as $|x|$ increases.

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \rightarrow \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

EXAMPLE 3 These examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} &= \lim_{x \rightarrow \infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)} && \text{Divide numerator and denominator by } x^2. \\ &= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3} && \text{See Fig. 2.37.} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow -\infty} \frac{11x + 2}{2x^3 - 1} &= \lim_{x \rightarrow -\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)} && \text{Divide numerator and denominator by } x^3. \\ &= \frac{0 + 0}{2 - 0} = 0 && \text{See Fig. 2.38.} \end{aligned}$$

Cases for which the degree of the numerator is greater than the degree of the denominator are illustrated in Examples 10 and 14.

Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an *asymptote* of the graph.

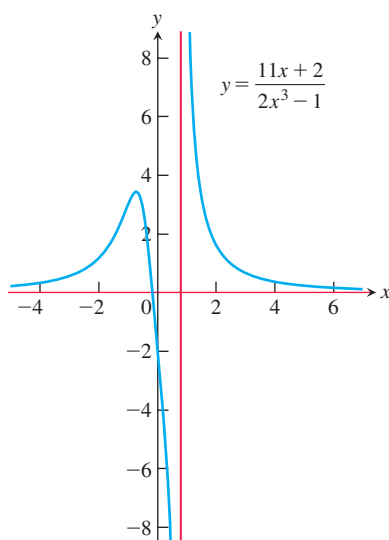


FIGURE 2.38 The graph of the function in Example 3b. The graph approaches the x -axis as $|x|$ increases.

Looking at $f(x) = 1/x$ (see Figure 2.35), we observe that the x -axis is an asymptote of the curve on the right because

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

and on the left because

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

We say that the x -axis is a *horizontal asymptote* of the graph of $f(x) = 1/x$.

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$

The graph of a function can have zero, one, or two horizontal asymptotes, depending on whether the function has limits as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

The graph of the function

$$f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

sketched in Figure 2.37 (Example 3a) has the line $y = 5/3$ as a horizontal asymptote on both the right and the left because

$$\lim_{x \rightarrow \infty} f(x) = \frac{5}{3} \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = \frac{5}{3}.$$

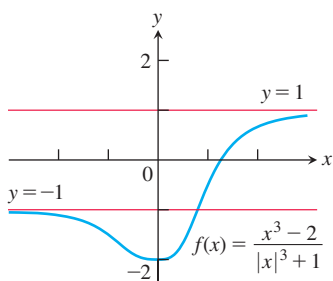


FIGURE 2.39 The graph of the function in Example 4 has two horizontal asymptotes.

EXAMPLE 4 Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

Solution We calculate the limits as $x \rightarrow \pm\infty$.

$$\text{For } x > 0: \quad \lim_{x \rightarrow \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow \infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x \rightarrow \infty} \frac{1 - (2/x^3)}{1 + (1/x^3)} = 1.$$

$$\text{For } x < 0: \quad \lim_{x \rightarrow -\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow -\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x \rightarrow -\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$$

The horizontal asymptotes are $y = -1$ and $y = 1$. The graph is displayed in Figure 2.39. Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x . ■

EXAMPLE 5 The x -axis (the line $y = 0$) is a horizontal asymptote of the graph of $y = e^x$ because

$$\lim_{x \rightarrow -\infty} e^x = 0.$$

To see this, we use the definition of a limit as x approaches $-\infty$. So let $\varepsilon > 0$ be given, but arbitrary. We must find a constant N such that

$$|e^x - 0| < \varepsilon \quad \text{whenever} \quad x < N.$$

Now $|e^x - 0| = e^x$, so the condition that needs to be satisfied whenever $x < N$ is

$$e^x < \varepsilon.$$

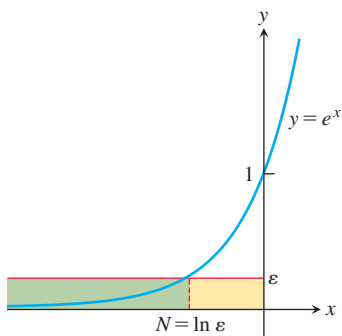


FIGURE 2.40 The graph of $y = e^x$ approaches the x -axis as $x \rightarrow -\infty$ (Example 5).

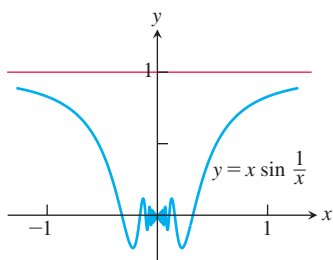


FIGURE 2.41 The line $y = 1$ is a horizontal asymptote of the function graphed here (Example 6b).

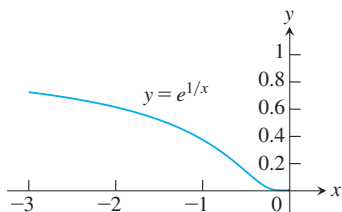


FIGURE 2.42 The graph of $y = e^{1/x}$ for $x < 0$ shows $\lim_{x \rightarrow 0^-} e^{1/x} = 0$ (Example 7).

Let $x = N$ be the number where $e^x = \varepsilon$. Since e^x is an increasing function, if $x < N$, then $e^x < \varepsilon$. We find N by taking the natural logarithm of both sides of the equation $e^N = \varepsilon$, so $N = \ln \varepsilon$ (see Figure 2.40). With this value of N the condition is satisfied, and we conclude that $\lim_{x \rightarrow -\infty} e^x = 0$. ■

Sometimes it is helpful to transform a limit in which x approaches ∞ to a new limit by setting $t = 1/x$ and seeing what happens as t approaches 0.

EXAMPLE 6 Find (a) $\lim_{x \rightarrow \infty} \sin(1/x)$ and (b) $\lim_{x \rightarrow \pm\infty} x \sin(1/x)$.

Solution

(a) We introduce the new variable $t = 1/x$. From Example 1, we know that $t \rightarrow 0^+$ as $x \rightarrow \infty$ (see Figure 2.35). Therefore,

$$\lim_{x \rightarrow \infty} \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \sin t = 0.$$

(b) We calculate the limits as $x \rightarrow \infty$ and $x \rightarrow -\infty$:

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^-} \frac{\sin t}{t} = 1.$$

The graph is shown in Figure 2.41, and we see that the line $y = 1$ is a horizontal asymptote. ■

Similarly, we can investigate the behavior of $y = f(1/x)$ as $x \rightarrow 0$ by investigating $y = f(t)$ as $t \rightarrow \pm\infty$, where $t = 1/x$.

EXAMPLE 7 Find $\lim_{x \rightarrow 0^-} e^{1/x}$.

Solution We let $t = 1/x$. From Figure 2.35, we can see that $t \rightarrow -\infty$ as $x \rightarrow 0^-$. (We make this idea more precise further on.) Therefore,

$$\lim_{x \rightarrow 0^-} e^{1/x} = \lim_{t \rightarrow -\infty} e^t = 0 \quad \text{Example 5}$$

(Figure 2.42). ■

The Sandwich Theorem also holds for limits as $x \rightarrow \pm\infty$. You must be sure, though, that the function whose limit you are trying to find stays between the bounding functions for all very large x in the positive direction (if $x \rightarrow \infty$) or all very large x in the negative direction (if $x \rightarrow -\infty$).

EXAMPLE 8 Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$. Since

$$0 \leq \left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|$$

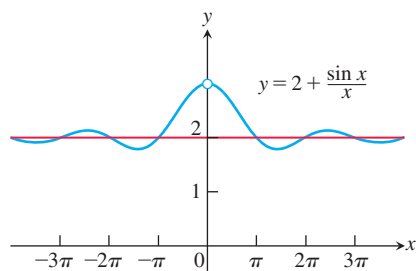


FIGURE 2.43 A curve may cross one of its asymptotes infinitely often (Example 8).

and $\lim_{x \rightarrow \pm\infty} |1/x| = 0$, we have $\lim_{x \rightarrow \pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x \rightarrow \pm\infty} \left(2 + \frac{\sin x}{x} \right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right (Figure 2.43).

This example illustrates that a curve may cross one of its horizontal asymptotes many times. ■

EXAMPLE 9 Find $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16})$.

Solution Both of the terms x and $\sqrt{x^2 + 16}$ approach infinity as $x \rightarrow \infty$, so what happens to the difference in the limit is unclear (we cannot subtract ∞ from ∞ because the symbol does not represent a real number). In this situation we can multiply the numerator and the denominator by the conjugate radical expression to obtain an equivalent algebraic expression:

$$\begin{aligned} \lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16}) &= \lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16}) \frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}} && \text{Multiply and divide by the conjugate.} \\ &= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}} = \lim_{x \rightarrow \infty} \frac{-16}{x + \sqrt{x^2 + 16}}. \end{aligned}$$

As $x \rightarrow \infty$, the denominator in this last expression becomes arbitrarily large, while the numerator remains constant, so we see that the limit is 0. We can also obtain this result by a direct calculation using the Limit Laws:

$$\lim_{x \rightarrow \infty} \frac{-16}{x + \sqrt{x^2 + 16}} = \lim_{x \rightarrow \infty} \frac{-16}{1 + \sqrt{\frac{x^2}{x^2} + \frac{16}{x^2}}} = \frac{0}{1 + \sqrt{1 + 0}} = 0. \quad \text{■}$$

Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \rightarrow \pm\infty$.

EXAMPLE 10 Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

in Figure 2.44.

Solution We are interested in the behavior as $x \rightarrow \pm\infty$. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$\begin{array}{r} \frac{x}{2} + 1 \\ 2x - 4 \overline{) x^2 + 0x - 3} \\ \underline{x^2 - 2x} \\ 2x - 3 \\ \underline{2x - 4} \\ 1 \end{array}$$

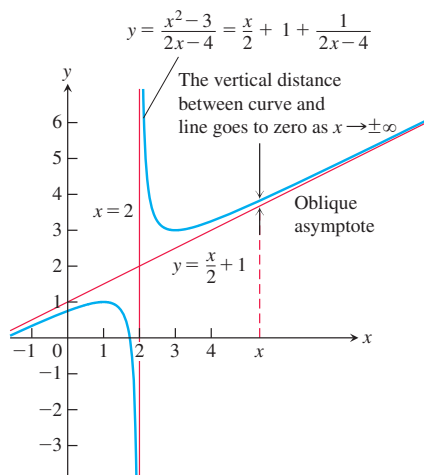


FIGURE 2.44 The graph of the function in Example 10 has an oblique asymptote.

This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\left(\frac{x}{2} + 1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x - 4}\right)}_{\text{remainder}}$$

As $x \rightarrow \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g , goes to zero, making the slanted line

$$g(x) = \frac{x}{2} + 1$$

an asymptote of the graph of f (Figure 2.44). The line $y = g(x)$ is an asymptote both to the right and to the left. ■

Notice in Example 10 that if the degree of the numerator in a rational function is greater than the degree of the denominator, then the limit as $|x|$ becomes large is $+\infty$ or $-\infty$, depending on the signs assumed by the numerator and denominator.

Infinite Limits

Let us look again at the function $f(x) = 1/x$. As $x \rightarrow 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B , however large, the values of f become larger still (Figure 2.45).

Thus, f has no limit as $x \rightarrow 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \rightarrow 0^+$. We write

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty.$$

In writing this equation, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞ , for there is no such number. Rather, this expression is just a concise way of saying that $\lim_{x \rightarrow 0^+} (1/x)$ *does not exist because $1/x$ becomes arbitrarily large and positive as $x \rightarrow 0^+$* .

As $x \rightarrow 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. (See Figure 2.45.) We write

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \rightarrow 0^-$ *does not exist because its values become arbitrarily large and negative*.

EXAMPLE 11 Find $\lim_{x \rightarrow 1^+} \frac{1}{x-1}$ and $\lim_{x \rightarrow 1^-} \frac{1}{x-1}$.

Geometric Solution The graph of $y = 1/(x-1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Figure 2.46). Therefore, $y = 1/(x-1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} = \infty \quad \text{and} \quad \lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty.$$

Analytic Solution Think about the number $x-1$ and its reciprocal. As $x \rightarrow 1^+$, we have $(x-1) \rightarrow 0^+$ and $1/(x-1) \rightarrow \infty$. As $x \rightarrow 1^-$, we have $(x-1) \rightarrow 0^-$ and $1/(x-1) \rightarrow -\infty$. ■

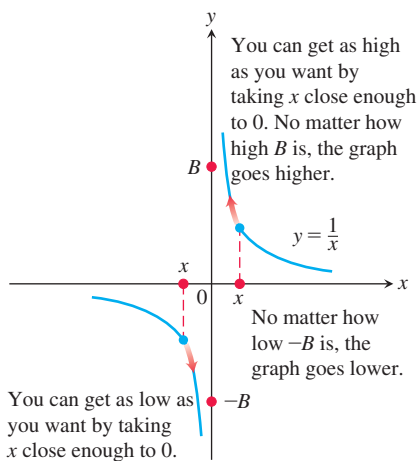


FIGURE 2.45 One-sided infinite limits:

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

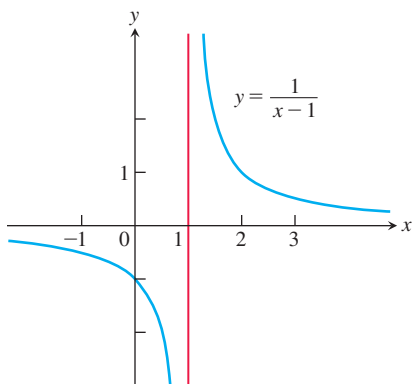


FIGURE 2.46 Near $x = 1$, the function $y = 1/(x-1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 11).

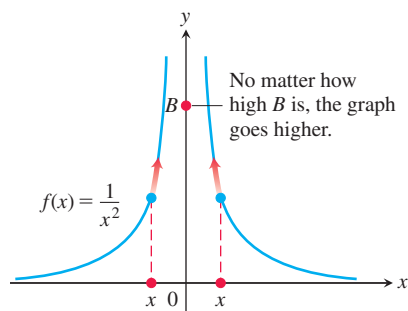


FIGURE 2.47 The graph of $f(x)$ in Example 12 approaches infinity as $x \rightarrow 0$.

EXAMPLE 12 Discuss the behavior of

$$f(x) = \frac{1}{x^2} \quad \text{as } x \rightarrow 0.$$

Solution As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Figure 2.47). This means that

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1}{x^2} = \infty.$$

The function $y = 1/x$ shows no consistent behavior as $x \rightarrow 0$. We have $1/x \rightarrow \infty$ if $x \rightarrow 0^+$, but $1/x \rightarrow -\infty$ if $x \rightarrow 0^-$. All we can say about $\lim_{x \rightarrow 0} (1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x \rightarrow 0} (1/x^2) = \infty$. ■

EXAMPLE 13 These examples illustrate that rational functions can behave in various ways near zeros of the denominator.

$$(a) \lim_{x \rightarrow 2} \frac{(x-2)^2}{x^2-4} = \lim_{x \rightarrow 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{x-2}{x+2} = 0$$

Can substitute 2 for x after algebraic manipulation eliminates division by 0.

$$(b) \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{4}$$

Again substitute 2 for x after algebraic manipulation eliminates division by 0.

$$(c) \lim_{x \rightarrow 2^+} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$$

The values are negative for $x > 2$, x near 2.

$$(d) \lim_{x \rightarrow 2^-} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$$

The values are positive for $x < 2$, x near 2.

$$(e) \lim_{x \rightarrow 2} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-3}{(x-2)(x+2)} \text{ does not exist.}$$

Limits from left and from right differ.

$$(f) \lim_{x \rightarrow 2} \frac{2-x}{(x-2)^3} = \lim_{x \rightarrow 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x \rightarrow 2} \frac{-1}{(x-2)^2} = -\infty$$

Denominator is positive, so values are negative near $x = 2$.

In parts (a) and (b), the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero factor in the denominator. ■

EXAMPLE 14 Find $\lim_{x \rightarrow -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7}$.

Solution We are asked to find the limit of a rational function as $x \rightarrow -\infty$, so we divide the numerator and denominator by x^2 , the highest power of x in the denominator:

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7} &= \lim_{x \rightarrow -\infty} \frac{2x^3 - 6x^2 + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\ &= \lim_{x \rightarrow -\infty} \frac{2x^2(x-3) + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\ &= -\infty, \end{aligned} \quad x^{-n} \rightarrow 0, x-3 \rightarrow -\infty$$

because the numerator tends to $-\infty$ while the denominator approaches 3 as $x \rightarrow -\infty$. ■

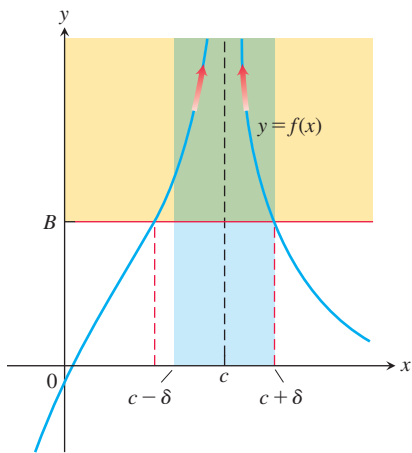


FIGURE 2.48 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies above the line $y = B$.

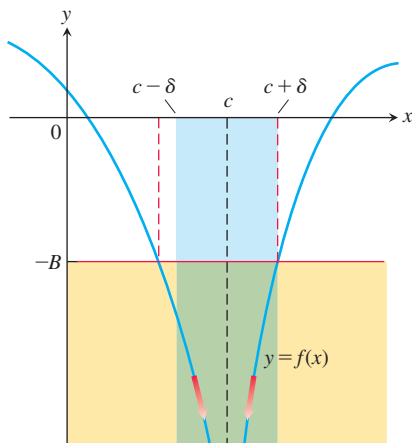


FIGURE 2.49 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

Precise Definitions of Infinite Limits

Instead of requiring $f(x)$ to lie arbitrarily close to a finite number L for all x sufficiently close to c , the definitions of infinite limits require $f(x)$ to lie arbitrarily far from zero. Except for this change, the language is very similar to what we have seen before. Figures 2.48 and 2.49 accompany these definitions.

DEFINITIONS

1. We say that $f(x)$ **approaches infinity as x approaches c** , and write

$$\lim_{x \rightarrow c} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that

$$f(x) > B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

2. We say that $f(x)$ **approaches negative infinity as x approaches c** , and write

$$\lim_{x \rightarrow c} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that

$$f(x) < -B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

The precise definitions of one-sided infinite limits at c are similar and are stated in the exercises.

EXAMPLE 15 Prove that $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$.

Solution Given $B > 0$, we want to find $\delta > 0$ such that

$$\frac{1}{x^2} > B \quad \text{whenever} \quad 0 < |x - 0| < \delta.$$

Now,

$$\frac{1}{x^2} > B \quad \text{if and only if} \quad x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$\text{if } 0 < |x| < \delta \quad \text{then} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \geq B.$$

Therefore, by definition,

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty.$$



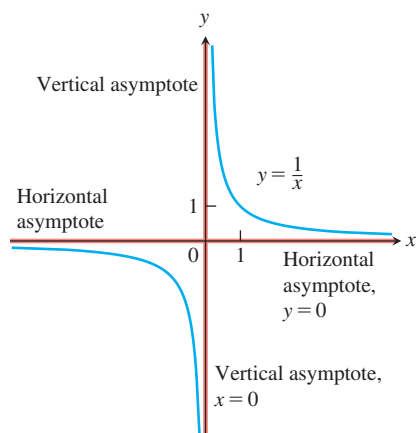


FIGURE 2.50 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

Vertical Asymptotes

Notice that the distance between a point on the graph of $f(x) = 1/x$ and the y -axis approaches zero as the point moves nearly vertically along the graph and away from the origin (Figure 2.50). The function $f(x) = 1/x$ is unbounded as x approaches 0 because

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

We say that the line $x = 0$ (the y -axis) is a *vertical asymptote* of the graph of $f(x) = 1/x$. Observe that the denominator is zero at $x = 0$ and the function is undefined there.

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm\infty.$$

EXAMPLE 16 Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x+3}{x+2}.$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$ and the behavior as $x \rightarrow -2$, where the denominator is zero.

The asymptotes are revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x+3)$ into $(x+2)$:

$$\begin{array}{r} x+2 \overline{) x+3} \\ \underline{x+2} \\ 1 \end{array}$$

This result enables us to rewrite y as

$$y = 1 + \frac{1}{x+2}.$$

We see that the curve in question is the graph of $f(x) = 1/x$ shifted 1 unit up and 2 units left (Figure 2.51). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$. As $x \rightarrow \pm\infty$, the curve approaches the horizontal asymptote $y = 1$; as $x \rightarrow -2$, the curve approaches the vertical asymptote $x = -2$. ■

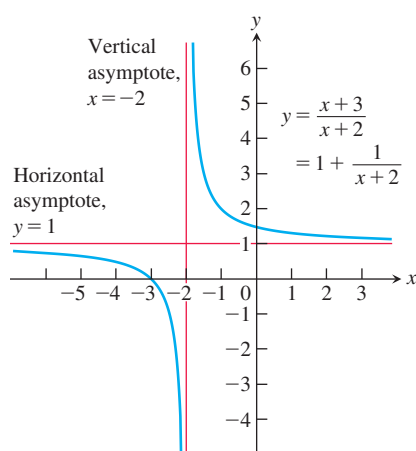


FIGURE 2.51 The lines $y = 1$ and $x = -2$ are asymptotes of the curve in Example 16.

EXAMPLE 17 Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$ and as $x \rightarrow \pm 2$, where the denominator is zero. Notice that f is an even function of x , so its graph is symmetric with respect to the y -axis.

(a) *The behavior as $x \rightarrow \pm\infty$.* Since $\lim_{x \rightarrow \infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well

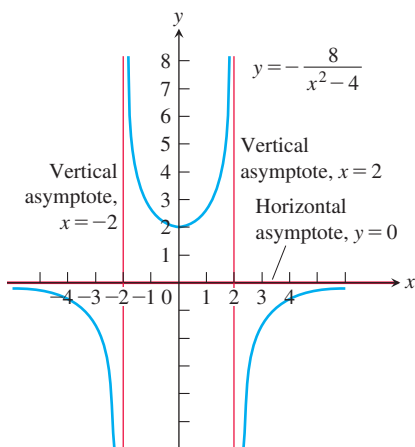


FIGURE 2.52 Graph of the function in Example 17. Notice that the curve approaches the x -axis from only one side. Asymptotes do not have to be two-sided.

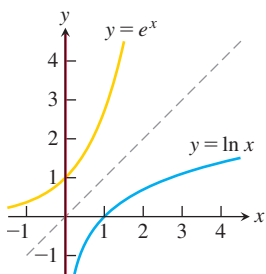


FIGURE 2.53 The vertical line $x = 0$ is a vertical asymptote of the natural logarithm function (Example 18).

(Figure 2.52). Notice that the curve approaches the x -axis from only the negative side (or from below). Also, $f(0) = 2$.

(b) The behavior as $x \rightarrow \pm 2$. Since

$$\lim_{x \rightarrow 2^+} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

There are no other asymptotes because f has a finite limit at all other points. ■

EXAMPLE 18 The graph of the natural logarithm function has the y -axis (the line $x = 0$) as a vertical asymptote. We see this from the graph sketched in Figure 2.53 (which is the reflection of the graph of the natural exponential function across the line $y = x$) and the fact that the x -axis is a horizontal asymptote of $y = e^x$ (Example 5). Thus,

$$\lim_{x \rightarrow 0^+} \ln x = -\infty.$$

The same result is true for $y = \log_a x$ whenever $a > 1$. ■

EXAMPLE 19 The curves

$$y = \sec x = \frac{1}{\cos x} \quad \text{and} \quad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, which are the points where $\cos x = 0$ (Figure 2.54).

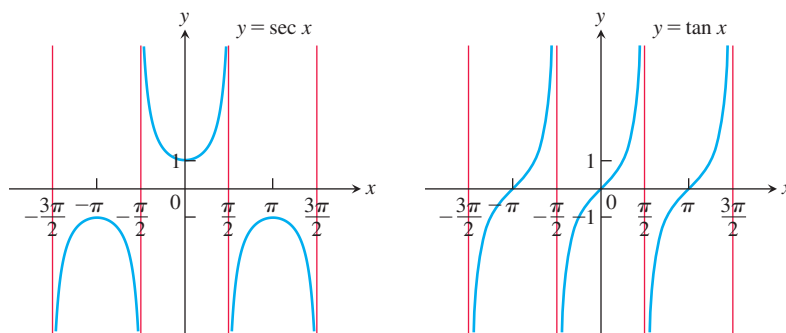


FIGURE 2.54 The graphs of $\sec x$ and $\tan x$ have infinitely many vertical asymptotes (Example 19). ■

Dominant Terms

In Example 10 we saw that by using long division, we can rewrite the function

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

as a linear function plus a remainder term:

$$f(x) = \left(\frac{x}{2} + 1\right) + \left(\frac{1}{2x - 4}\right).$$

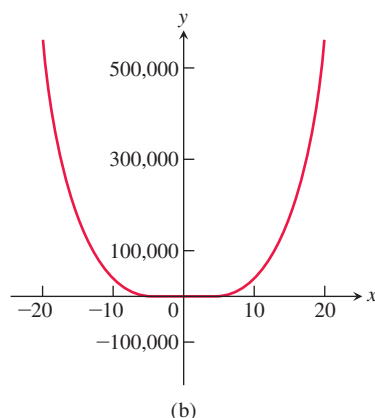
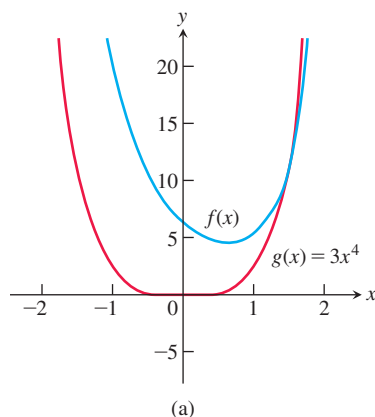


FIGURE 2.55 The graphs of f and g are (a) distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 20).

This tells us immediately that

For x large: $f(x) \approx \frac{x}{2} + 1$ $\frac{1}{2x-4}$ is near 0.

For x near 2: $f(x) \approx \frac{1}{2x-4}$ This term is very large in absolute value.

If we want to know how f behaves, this is one possible way to find out. It behaves like $y = (x/2) + 1$ when $|x|$ is large and the contribution of $1/(2x-4)$ to the total value of f is insignificant. It behaves like $1/(2x-4)$ when x is so close to 2 that $1/(2x-4)$ makes the dominant contribution.

We say that $(x/2) + 1$ **dominates** when x approaches ∞ or $-\infty$, and we say that $1/(2x-4)$ dominates when x approaches 2. **Dominant terms** like these help us predict a function's behavior.

EXAMPLE 20 Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that although f and g are quite different for numerically small values of x , they behave similarly for very large $|x|$, in the sense that their ratios approach 1 as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

Solution The graphs of f and g behave quite differently near the origin (Figure 2.55a), but appear as virtually identical on a larger scale (Figure 2.55b).

We can test that the term $3x^4$ in f , represented graphically by g , dominates the polynomial f for numerically large values of x by examining the ratio of the two functions as $x \rightarrow \pm\infty$. We find that

$$\begin{aligned} \lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\ &= \lim_{x \rightarrow \pm\infty} \left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4} \right) \\ &= 1, \end{aligned}$$

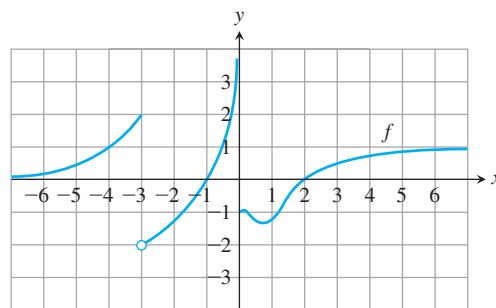
which means that f and g appear nearly identical when $|x|$ is large. ■

EXERCISES 2.5

Finding Limits

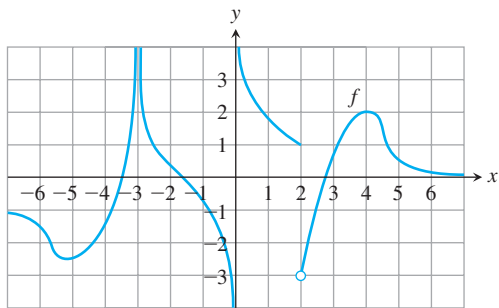
1. For the function f whose graph is given, determine the following limits. Write ∞ or $-\infty$ where appropriate.

- | | |
|--|---------------------------------------|
| a. $\lim_{x \rightarrow 2} f(x)$ | b. $\lim_{x \rightarrow -3^+} f(x)$ |
| c. $\lim_{x \rightarrow -3^-} f(x)$ | d. $\lim_{x \rightarrow -3} f(x)$ |
| e. $\lim_{x \rightarrow 0^+} f(x)$ | f. $\lim_{x \rightarrow 0^-} f(x)$ |
| g. $\lim_{x \rightarrow 0} f(x)$ | h. $\lim_{x \rightarrow \infty} f(x)$ |
| i. $\lim_{x \rightarrow -\infty} f(x)$ | |



2. For the function f whose graph is given, determine the following limits. Write ∞ or $-\infty$ where appropriate.

- | | | |
|-----------------------------------|---------------------------------------|--|
| a. $\lim_{x \rightarrow 4} f(x)$ | b. $\lim_{x \rightarrow 2^+} f(x)$ | c. $\lim_{x \rightarrow 2^-} f(x)$ |
| d. $\lim_{x \rightarrow 2} f(x)$ | e. $\lim_{x \rightarrow -3^+} f(x)$ | f. $\lim_{x \rightarrow -3^-} f(x)$ |
| g. $\lim_{x \rightarrow -3} f(x)$ | h. $\lim_{x \rightarrow 0^+} f(x)$ | i. $\lim_{x \rightarrow 0^-} f(x)$ |
| j. $\lim_{x \rightarrow 0} f(x)$ | k. $\lim_{x \rightarrow \infty} f(x)$ | l. $\lim_{x \rightarrow -\infty} f(x)$ |



In Exercises 3–8, find the limit of each function (a) as $x \rightarrow \infty$ and (b) as $x \rightarrow -\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

- | | |
|--|--|
| 3. $f(x) = \frac{2}{x} - 3$ | 4. $f(x) = \pi - \frac{2}{x^2}$ |
| 5. $g(x) = \frac{1}{2 + (1/x)}$ | 6. $g(x) = \frac{1}{8 - (5/x^2)}$ |
| 7. $h(x) = \frac{-5 + (7/x)}{3 - (1/x^2)}$ | 8. $h(x) = \frac{3 - (2/x)}{4 + (\sqrt{2}/x^2)}$ |

Find the limits in Exercises 9–12.

- | | |
|--|--|
| 9. $\lim_{x \rightarrow \infty} \frac{\sin 2x}{x}$ | 10. $\lim_{\theta \rightarrow -\infty} \frac{\cos \theta}{3\theta}$ |
| 11. $\lim_{t \rightarrow -\infty} \frac{2 - t + \sin t}{t + \cos t}$ | 12. $\lim_{r \rightarrow \infty} \frac{r + \sin r}{2r + 7 - 5 \sin r}$ |

Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function (a) as $x \rightarrow \infty$ and (b) as $x \rightarrow -\infty$. Write ∞ or $-\infty$ where appropriate.

- | | |
|--|---|
| 13. $f(x) = \frac{2x + 3}{5x + 7}$ | 14. $f(x) = \frac{2x^3 + 7}{x^3 - x^2 + x + 7}$ |
| 15. $f(x) = \frac{x + 1}{x^2 + 3}$ | 16. $f(x) = \frac{3x + 7}{x^2 - 2}$ |
| 17. $h(x) = \frac{7x^3}{x^3 - 3x^2 + 6x}$ | 18. $h(x) = \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$ |
| 19. $g(x) = \frac{10x^5 + x^4 + 31}{x^6}$ | 20. $g(x) = \frac{x^3 + 7x^2 - 2}{x^2 - x + 1}$ |
| 21. $f(x) = \frac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3}$ | 22. $h(x) = \frac{5x^8 - 2x^3 + 9}{3 + x - 4x^5}$ |

Limits as $x \rightarrow \infty$ or $x \rightarrow -\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x .

Divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 23–36. Write ∞ or $-\infty$ where appropriate.

- | | |
|--|--|
| 23. $\lim_{x \rightarrow \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}}$ | 24. $\lim_{x \rightarrow -\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3}$ |
| 25. $\lim_{x \rightarrow -\infty} \left(\frac{1 - x^3}{x^2 + 7x} \right)^5$ | 26. $\lim_{x \rightarrow \infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}}$ |
| 27. $\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$ | 28. $\lim_{x \rightarrow \infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$ |
| 29. $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$ | 30. $\lim_{x \rightarrow \infty} \frac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$ |
| 31. $\lim_{x \rightarrow \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$ | 32. $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$ |
| 33. $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x + 1}$ | 34. $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$ |
| 35. $\lim_{x \rightarrow \infty} \frac{x - 3}{\sqrt{4x^2 + 25}}$ | 36. $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$ |

Infinite Limits

Find the limits in Exercises 37–48. Write ∞ or $-\infty$ where appropriate.

- | | |
|--|--|
| 37. $\lim_{x \rightarrow 0^+} \frac{1}{3x}$ | 38. $\lim_{x \rightarrow 0^-} \frac{5}{2x}$ |
| 39. $\lim_{x \rightarrow 2^-} \frac{3}{x - 2}$ | 40. $\lim_{x \rightarrow 3^+} \frac{1}{x - 3}$ |
| 41. $\lim_{x \rightarrow -8^+} \frac{2x}{x + 8}$ | 42. $\lim_{x \rightarrow -5^-} \frac{3x}{2x + 10}$ |
| 43. $\lim_{x \rightarrow 7} \frac{4}{(x - 7)^2}$ | 44. $\lim_{x \rightarrow 0} \frac{-1}{x^2(x + 1)}$ |
| 45. a. $\lim_{x \rightarrow 0^+} \frac{2}{3x^{1/3}}$ | b. $\lim_{x \rightarrow 0^-} \frac{2}{3x^{1/3}}$ |
| 46. a. $\lim_{x \rightarrow 0^+} \frac{2}{x^{1/5}}$ | b. $\lim_{x \rightarrow 0^-} \frac{2}{x^{1/5}}$ |
| 47. $\lim_{x \rightarrow 0} \frac{4}{x^{2/5}}$ | 48. $\lim_{x \rightarrow 0} \frac{1}{x^{2/3}}$ |

Find the limits in Exercises 49–52. Write ∞ or $-\infty$ where appropriate.

- | | |
|---|---|
| 49. $\lim_{x \rightarrow (\pi/2)^-} \tan x$ | 50. $\lim_{x \rightarrow (-\pi/2)^+} \sec x$ |
| 51. $\lim_{\theta \rightarrow 0^-} (1 + \csc \theta)$ | 52. $\lim_{\theta \rightarrow 0} (2 - \cot \theta)$ |

Find the limits in Exercises 53–58. Write ∞ or $-\infty$ where appropriate.

- | | | |
|---|-------------------------|-------------------------|
| 53. $\lim_{x \rightarrow 2} \frac{1}{x^2 - 4}$ as | a. $x \rightarrow 2^+$ | b. $x \rightarrow 2^-$ |
| | c. $x \rightarrow -2^+$ | d. $x \rightarrow -2^-$ |
| 54. $\lim_{x \rightarrow 1} \frac{x}{x^2 - 1}$ as | a. $x \rightarrow 1^+$ | b. $x \rightarrow 1^-$ |
| | c. $x \rightarrow -1^+$ | d. $x \rightarrow -1^-$ |

55. $\lim \left(\frac{x^2}{2} - \frac{1}{x} \right)$ as

a. $x \rightarrow 0^+$

b. $x \rightarrow 0^-$

c. $x \rightarrow \sqrt[3]{2}$

d. $x \rightarrow -1$

56. $\lim \frac{x^2 - 1}{2x + 4}$ as

a. $x \rightarrow -2^+$

b. $x \rightarrow -2^-$

c. $x \rightarrow 1^+$

d. $x \rightarrow 0^-$

57. $\lim \frac{x^2 - 3x + 2}{x^3 - 2x^2}$ as

a. $x \rightarrow 0^+$

b. $x \rightarrow 2^+$

c. $x \rightarrow 2^-$

d. $x \rightarrow 2$

e. What, if anything, can be said about the limit as $x \rightarrow 0$?

58. $\lim \frac{x^2 - 3x + 2}{x^3 - 4x}$ as

a. $x \rightarrow 2^+$

b. $x \rightarrow -2^+$

c. $x \rightarrow 0^-$

d. $x \rightarrow 1^+$

e. What, if anything, can be said about the limit as $x \rightarrow 0$?Find the limits in Exercises 59–62. Write ∞ or $-\infty$ where appropriate.

59. $\lim \left(2 - \frac{3}{t^{1/3}} \right)$ as

a. $t \rightarrow 0^+$

b. $t \rightarrow 0^-$

60. $\lim \left(\frac{1}{t^{3/5}} + 7 \right)$ as

a. $t \rightarrow 0^+$

b. $t \rightarrow 0^-$

61. $\lim \left(\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}} \right)$ as

a. $x \rightarrow 0^+$

b. $x \rightarrow 0^-$

c. $x \rightarrow 1^+$

d. $x \rightarrow 1^-$

62. $\lim \left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}} \right)$ as

a. $x \rightarrow 0^+$

b. $x \rightarrow 0^-$

c. $x \rightarrow 1^+$

d. $x \rightarrow 1^-$

Graphing Simple Rational Functions

Graph the rational functions in Exercises 63–68. Include the graphs and equations of the asymptotes and dominant terms.

63. $y = \frac{1}{x-1}$

64. $y = \frac{1}{x+1}$

65. $y = \frac{1}{2x+4}$

66. $y = \frac{-3}{x-3}$

67. $y = \frac{x+3}{x+2}$

68. $y = \frac{2x}{x+1}$

Domains and Asymptotes

Determine the domain of each function in Exercises 69–74. Then use various limits to find the asymptotes.

69. $y = 4 + \frac{3x^2}{x^2 + 1}$

70. $y = \frac{2x}{x^2 - 1}$

71. $y = \frac{8 - e^x}{2 + e^x}$

72. $y = \frac{4e^x + e^{2x}}{e^x + e^{2x}}$

73. $y = \frac{\sqrt{x^2 + 4}}{x}$

74. $y = \frac{x^3}{x^3 - 8}$

Inventing Graphs and FunctionsIn Exercises 75–78, sketch the graph of a function $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)

75. $f(0) = 0, f(1) = 2, f(-1) = -2, \lim_{x \rightarrow -\infty} f(x) = -1$, and $\lim_{x \rightarrow \infty} f(x) = 1$

76. $f(0) = 0, \lim_{x \rightarrow \pm\infty} f(x) = 0, \lim_{x \rightarrow 0^+} f(x) = 2$, and $\lim_{x \rightarrow 0^-} f(x) = -2$

77. $f(0) = 0, \lim_{x \rightarrow \pm\infty} f(x) = 0, \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow -1^+} f(x) = \infty$, $\lim_{x \rightarrow 1^+} f(x) = -\infty$, and $\lim_{x \rightarrow -1^-} f(x) = -\infty$

78. $f(2) = 1, f(-1) = 0, \lim_{x \rightarrow \infty} f(x) = 0, \lim_{x \rightarrow 0^+} f(x) = \infty$, $\lim_{x \rightarrow 0^-} f(x) = -\infty$, and $\lim_{x \rightarrow -\infty} f(x) = 1$

In Exercises 79–82, find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

79. $\lim_{x \rightarrow \pm\infty} f(x) = 0, \lim_{x \rightarrow 2^-} f(x) = \infty$, and $\lim_{x \rightarrow 2^+} f(x) = \infty$

80. $\lim_{x \rightarrow \pm\infty} g(x) = 0, \lim_{x \rightarrow 3^-} g(x) = -\infty$, and $\lim_{x \rightarrow 3^+} g(x) = \infty$

81. $\lim_{x \rightarrow -\infty} h(x) = -1, \lim_{x \rightarrow \infty} h(x) = 1, \lim_{x \rightarrow 0^-} h(x) = -1$, and $\lim_{x \rightarrow 0^+} h(x) = 1$

82. $\lim_{x \rightarrow \pm\infty} k(x) = 1, \lim_{x \rightarrow 1^-} k(x) = \infty$, and $\lim_{x \rightarrow 1^+} k(x) = -\infty$

83. Suppose that $f(x)$ and $g(x)$ are polynomials in x and that $\lim_{x \rightarrow \infty} (f(x)/g(x)) = 2$. Can you conclude anything about $\lim_{x \rightarrow -\infty} (f(x)/g(x))$? Give reasons for your answer.

84. Suppose that $f(x)$ and $g(x)$ are polynomials in x . Can the graph of $f(x)/g(x)$ have an asymptote if $g(x)$ is never zero? Give reasons for your answer.

85. How many horizontal asymptotes can the graph of a given rational function have? Give reasons for your answer.

Finding Limits of Differences When $x \rightarrow \pm\infty$

Find the limits in Exercises 86–92. (Hint: Try multiplying and dividing by the conjugate.)

86. $\lim_{x \rightarrow \infty} (\sqrt{x+9} - \sqrt{x+4})$

87. $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 25} - \sqrt{x^2 - 1})$

88. $\lim_{x \rightarrow -\infty} (\sqrt{x^2 + 3} + x)$

89. $\lim_{x \rightarrow -\infty} (2x + \sqrt{4x^2 + 3x - 2})$

90. $\lim_{x \rightarrow \infty} (\sqrt{9x^2 - x} - 3x)$

$$91. \lim_{x \rightarrow \infty} (\sqrt{x^2 + 3x} - \sqrt{x^2 - 2x})$$

$$92. \lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - \sqrt{x^2 - x})$$

Using the Formal Definitions

Use the formal definitions of limits as $x \rightarrow \pm\infty$ to establish the limits in Exercises 93 and 94.

93. If f has the constant value $f(x) = k$, then $\lim_{x \rightarrow \infty} f(x) = k$.

94. If f has the constant value $f(x) = k$, then $\lim_{x \rightarrow -\infty} f(x) = k$.

Use formal definitions to prove the limit statements in Exercises 95–98.

$$95. \lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty$$

$$96. \lim_{x \rightarrow 0} \frac{1}{|x|} = \infty$$

$$97. \lim_{x \rightarrow 3} \frac{-2}{(x-3)^2} = -\infty$$

$$98. \lim_{x \rightarrow -5} \frac{1}{(x+5)^2} = \infty$$

99. Here is the definition of **infinite right-hand limit**.

Suppose that an interval (c, d) lies in the domain of f . We say that $f(x)$ approaches infinity as x approaches c from the right, and write

$$\lim_{x \rightarrow c^+} f(x) = \infty,$$

if, for every positive real number B , there exists a corresponding number $\delta > 0$ such that

$$f(x) > B \quad \text{whenever} \quad c < x < c + \delta.$$

Modify the definition to cover the following cases.

a. $\lim_{x \rightarrow c^-} f(x) = \infty$

b. $\lim_{x \rightarrow c^+} f(x) = -\infty$

c. $\lim_{x \rightarrow c^-} f(x) = -\infty$

Use the formal definitions from Exercise 99 to prove the limit statements in Exercises 100–104.

$$100. \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$$

$$101. \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

$$102. \lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty$$

$$103. \lim_{x \rightarrow 2^+} \frac{1}{x-2} = \infty$$

$$104. \lim_{x \rightarrow 1^-} \frac{1}{1-x^2} = \infty$$

Oblique Asymptotes

Graph the rational functions in Exercises 105–110. Include the graphs and equations of the asymptotes.

$$105. y = \frac{x^2}{x-1}$$

$$106. y = \frac{x^2+1}{x-1}$$

$$107. y = \frac{x^2-4}{x-1}$$

$$108. y = \frac{x^2-1}{2x+4}$$

$$109. y = \frac{x^2-1}{x}$$

$$110. y = \frac{x^3+1}{x^2}$$

Additional Graphing Exercises

T Graph the curves in Exercises 111–114. Explain the relationship between the curve's formula and what you see.

$$111. y = \frac{x}{\sqrt{4-x^2}}$$

$$112. y = \frac{-1}{\sqrt{4-x^2}}$$

$$113. y = x^{2/3} + \frac{1}{x^{1/3}}$$

$$114. y = \sin\left(\frac{\pi}{x^2+1}\right)$$

T Graph the functions in Exercises 115 and 116. Then answer the following questions.

a. How does the graph behave as $x \rightarrow 0^+$?

b. How does the graph behave as $x \rightarrow \pm\infty$?

c. How does the graph behave near $x = 1$ and $x = -1$?

Give reasons for your answers.

$$115. y = \frac{3}{2}\left(x - \frac{1}{x}\right)^{2/3}$$

$$116. y = \frac{3}{2}\left(\frac{x}{x-1}\right)^{2/3}$$

2.6 Continuity

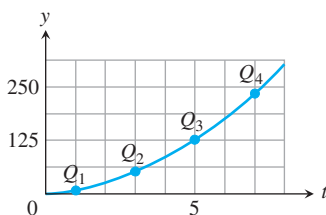


FIGURE 2.56 Connecting plotted points.

When we plot function values generated in a laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the points we did not measure (Figure 2.56). In doing so, we are assuming that we are working with a *continuous function*, so its outputs vary regularly and consistently with the inputs, and do not jump abruptly from one value to another without taking on the values in between. Intuitively, any function $y = f(x)$ whose graph can be sketched over its domain in one unbroken motion is an example of a continuous function. Such functions play an important role in the study of calculus and its applications.

Continuity at a Point

To understand continuity, it helps to consider a function like that in Figure 2.57, whose limits we investigated in Example 1 in the last section.

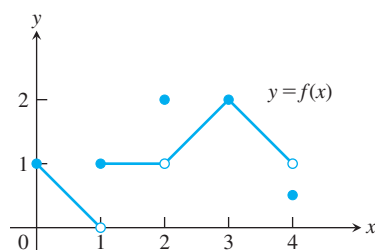


FIGURE 2.57 The function is not continuous at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

EXAMPLE 1 At which numbers does the function f in Figure 2.57 appear to be not continuous? Explain why. What occurs at other numbers in the domain?

Solution First we observe that the domain of the function is the closed interval $[0, 4]$, so we will be considering the numbers x within that interval. From the figure, we notice right away that there are breaks in the graph at the numbers $x = 1$, $x = 2$, and $x = 4$. The break at $x = 1$ appears as a jump, which we identify later as a “jump discontinuity.” The break at $x = 2$ is called a “removable discontinuity” since by changing the function definition at that one point, we can create a new function that is continuous at $x = 2$. Similarly, $x = 4$ is a removable discontinuity.

Numbers at which the graph of f has breaks:

At the interior point $x = 1$, the function fails to have a limit. It does have both a left-hand limit, $\lim_{x \rightarrow 1^-} f(x) = 0$, as well as a right-hand limit, $\lim_{x \rightarrow 1^+} f(x) = 1$, but the limit values are different, resulting in a jump in the graph. The function is not continuous at $x = 1$. However, the function value $f(1) = 1$ is equal to the limit from the right, so the function is continuous from the right at $x = 1$.

At $x = 2$, the function does have a limit, $\lim_{x \rightarrow 2} f(x) = 1$, but the value of the function is $f(2) = 2$. The limit and function values are not the same, so there is a break in the graph and f is not continuous at $x = 2$.

At $x = 4$, the function does have a left-hand limit at this right endpoint, $\lim_{x \rightarrow 4^-} f(x) = 1$, but again the value of the function $f(4) = \frac{1}{2}$ differs from the value of the limit. We see again a break in the graph of the function at this endpoint and the function is not continuous from the left.

Numbers at which the graph of f has no breaks:

At $x = 3$, the function has a limit, $\lim_{x \rightarrow 3} f(x) = 2$. Moreover, the limit is the same value as the function there, $f(3) = 2$. The function is continuous at $x = 3$.

At $x = 0$, the function has a right-hand limit at this left endpoint, $\lim_{x \rightarrow 0^+} f(x) = 1$, and the value of the function is the same, $f(0) = 1$. The function is continuous from the right at $x = 0$. Because $x = 0$ is a left endpoint of the function’s domain, we have that $\lim_{x \rightarrow 0} f(x) = 1$ and so f is continuous at $x = 0$.

At all other numbers $x = c$ in the domain, the function has a limit equal to the value of the function, so $\lim_{x \rightarrow c} f(x) = f(c)$. For example, $\lim_{x \rightarrow 5/2} f(x) = f(5/2) = 3/2$. No breaks appear in the graph of the function at any of these numbers and the function is continuous at each of them. ■

The following definitions capture the continuity ideas we observed in Example 1.

DEFINITIONS Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f .

The function f is **continuous at c** if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

The function f is **right-continuous at c** (or **continuous from the right**) if

$$\lim_{x \rightarrow c^+} f(x) = f(c).$$

The function f is **left-continuous at c** (or **continuous from the left**) if

$$\lim_{x \rightarrow c^-} f(x) = f(c).$$

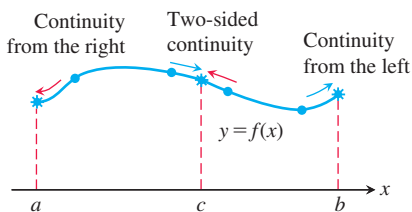


FIGURE 2.58 Continuity at points a , b , and c .

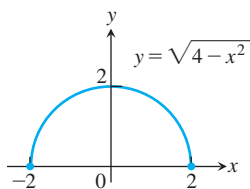


FIGURE 2.59 A function that is continuous over its domain (Example 2).

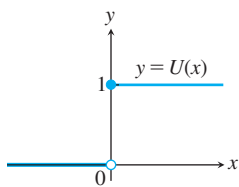


FIGURE 2.60 A function that has a jump discontinuity at the origin (Example 3).

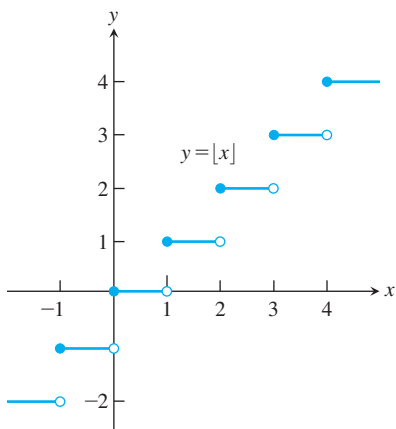


FIGURE 2.61 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

The function f in Example 1 is continuous at every x in $[0, 4]$ except $x = 1$, 2 , and 4 . It is right-continuous but not left-continuous at $x = 1$, neither right- nor left-continuous at $x = 2$, and not left-continuous at $x = 4$.

From Theorem 5, it follows immediately that a function f is continuous at an *interior* point c of an interval in its domain if and only if it is both right-continuous and left-continuous at c (Figure 2.58). We say that a function is **continuous over a closed interval** $[a, b]$ if it is right-continuous at a , left-continuous at b , and continuous at all interior points of the interval. This definition applies to the infinite closed intervals $[a, \infty)$ and $(-\infty, b]$ as well, but only one endpoint is involved. If a function is not continuous at point c of its domain, we say that f is **discontinuous at c** and that f has a discontinuity at c . Note that a function f can be continuous, right-continuous, or left-continuous only at a point c for which $f(c)$ is defined.

EXAMPLE 2 The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$ (Figure 2.59). It is continuous at all points of this interval, including the endpoints $x = -2$ and $x = 2$. ■

EXAMPLE 3 The unit step function $U(x)$, graphed in Figure 2.60, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$. ■

At an interior point or an endpoint of an interval in its domain, a function is continuous at points where it passes the following test.

Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \rightarrow c} f(x)$ exists (f has a limit as $x \rightarrow c$).
3. $\lim_{x \rightarrow c} f(x) = f(c)$ (the limit equals the function value).

For one-sided continuity, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

EXAMPLE 4 The function $y = \lfloor x \rfloor$ introduced in Section 1.1 is graphed in Figure 2.61. It is discontinuous at every integer n , because the left-hand and right-hand limits are not equal as $x \rightarrow n$:

$$\lim_{x \rightarrow n^-} \lfloor x \rfloor = n - 1 \quad \text{and} \quad \lim_{x \rightarrow n^+} \lfloor x \rfloor = n.$$

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

The greatest integer function is continuous at every real number other than the integers. For example,

$$\lim_{x \rightarrow 1.5} \lfloor x \rfloor = 1 = \lfloor 1.5 \rfloor.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x \rightarrow c} \lfloor x \rfloor = n - 1 = \lfloor c \rfloor. \quad \blacksquare$$

Figure 2.62 displays several common ways in which a function can fail to be continuous. The function in Figure 2.62a is continuous at $x = 0$. The function in Figure 2.62b does not contain $x = 0$ in its domain. It would be continuous if its domain were extended so

that $f(0) = 1$. The function in Figure 2.62c would be continuous if $f(0)$ were 1 instead of 2. The discontinuity in Figure 2.62c is **removable**. The function has a limit as $x \rightarrow 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in Figure 2.62d through f are more serious: $\lim_{x \rightarrow 0} f(x)$ does not exist, and there is no way to improve the situation by appropriately defining f at 0. The step function in Figure 2.62d has a **jump discontinuity**: The one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in Figure 2.62e has an **infinite discontinuity**. The function in Figure 2.62f has an **oscillating discontinuity**: It oscillates so much that its values approach each number in $[-1, 1]$ as $x \rightarrow 0$. Since it does not approach a single number, it does not have a limit as x approaches 0.

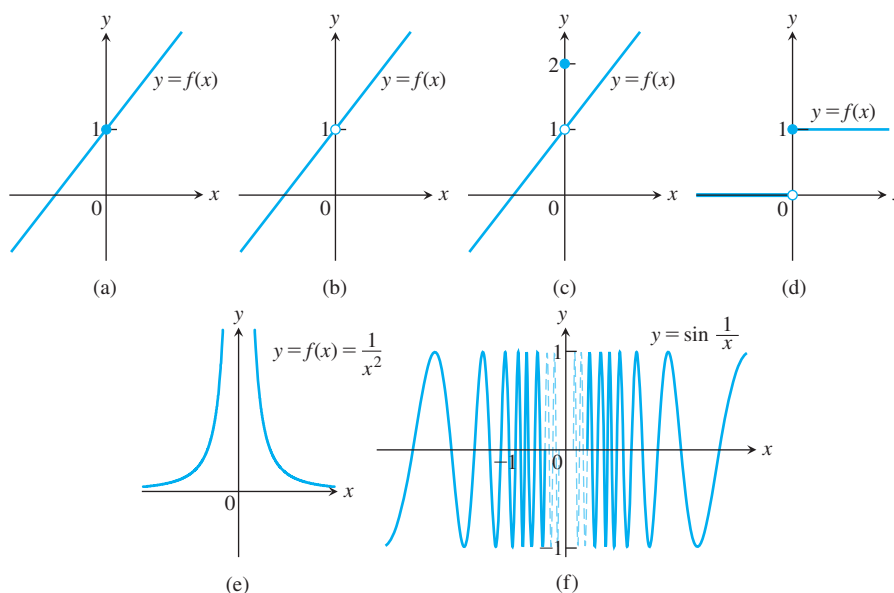


FIGURE 2.62 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Continuous Functions

We now describe the continuity behavior of a function throughout its entire domain, not only at a single point. We define a **continuous function** to be one that is continuous at every point in its domain. This is a property of the *function*. A function always has a specified domain, so if we change the domain, then we change the function, and this may change its continuity property as well. If a function is discontinuous at one or more points of its domain, we say it is a **discontinuous function**.

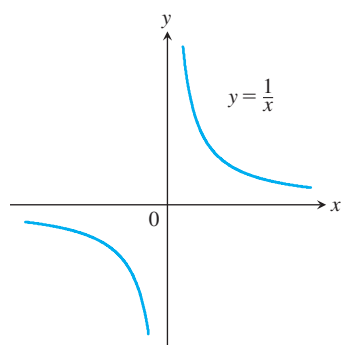


FIGURE 2.63 The function $f(x) = 1/x$ is continuous over its natural domain. It is not defined at the origin, so it is not continuous on any interval containing $x = 0$ (Example 5).

EXAMPLE 5

- (a) The function $f(x) = 1/x$ (Figure 2.63) is a continuous function because it is continuous at every point of its domain. The point $x = 0$ is not in the domain of the function f , so f is not continuous on any interval containing $x = 0$. Moreover, there is no way to extend f to a new function that is defined and continuous at $x = 0$. The function f does not have a removable discontinuity at $x = 0$.
- (b) The identity function $f(x) = x$ and constant functions are continuous everywhere by Example 3, Section 2.3. ■

Algebraic combinations of continuous functions are continuous wherever they are defined.

THEOREM 8—Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

- | | |
|-------------------------------|--|
| 1. <i>Sums:</i> | $f + g$ |
| 2. <i>Differences:</i> | $f - g$ |
| 3. <i>Constant multiples:</i> | $k \cdot f$, for any number k |
| 4. <i>Products:</i> | $f \cdot g$ |
| 5. <i>Quotients:</i> | f/g , provided $g(c) \neq 0$ |
| 6. <i>Powers:</i> | f^n , n a positive integer |
| 7. <i>Roots:</i> | $\sqrt[n]{f}$, provided it is defined on an interval containing c , where n is a positive integer |

Most of the results in Theorem 8 follow from the limit rules in Theorem 1, Section 2.2. For instance, to prove the sum property we have

$$\begin{aligned}
 \lim_{x \rightarrow c} (f + g)(x) &= \lim_{x \rightarrow c} (f(x) + g(x)) \\
 &= \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) && \text{Sum Rule, Theorem 1} \\
 &= f(c) + g(c) && \text{Continuity of } f, g \text{ at } c \\
 &= (f + g)(c).
 \end{aligned}$$

This shows that $f + g$ is continuous.

EXAMPLE 6

- (a) Every polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x \rightarrow c} P(x) = P(c)$ by Theorem 2, Section 2.2.
- (b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(c) \neq 0$) by Theorem 3, Section 2.2. ■

EXAMPLE 7 The function $f(x) = |x|$ is continuous. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x \rightarrow 0} |x| = 0 = |0|$. ■

The functions $y = \sin x$ and $y = \cos x$ are continuous at $x = 0$ by Example 12 of Section 2.2. Both functions are continuous everywhere (see Exercise 76). It follows from Theorem 8 that all six trigonometric functions are continuous wherever they are defined. For example, $y = \tan x$ is continuous on $\cdots \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \cdots$.

Inverse Functions and Continuity

When a continuous function defined on an interval has an inverse, the inverse function is itself a continuous function over its own domain. This result is suggested by the observation that the graph of f^{-1} , being the reflection of the graph of f across the line $y = x$, cannot have any breaks in it when the graph of f has no breaks. A rigorous proof that f^{-1} is continuous whenever f is continuous on an interval is given in more advanced texts. As an example, the inverse trigonometric functions are all continuous over their domains.

We defined the exponential function $y = a^x$ in Section 1.4 informally. The graph was obtained from the graph of $y = a^x$ for x , a rational number, by “filling in the holes” at the irrational points x , so as to make the function $y = a^x$ continuous over the entire real line. The inverse function $y = \log_a x$ is also continuous. In particular, the natural exponential function $y = e^x$ and the natural logarithm function $y = \ln x$ are both continuous over their domains. Proofs of continuity for these functions will be given in Chapter 7.

Continuity of Compositions of Functions

Functions obtained by composing continuous functions are continuous. If $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is also continuous at $x = c$ (Figure 2.64). In this case, the limit of $g \circ f$ as $x \rightarrow c$ is $g(f(c))$.

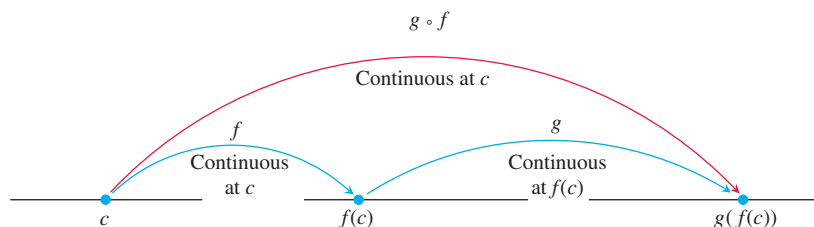


FIGURE 2.64 Compositions of continuous functions are continuous.

THEOREM 9—Compositions of Continuous Functions

If f is continuous at c , and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c .

Intuitively, Theorem 9 is reasonable because if x is close to c , then $f(x)$ is close to $f(c)$, and since g is continuous at $f(c)$, it follows that $g(f(x))$ is close to $g(f(c))$.

The continuity of compositions holds for any finite number of compositions of functions. The only requirement is that each function be continuous where it is applied. An outline of a proof of Theorem 9 is given in Exercise 6 in Appendix A.6.

EXAMPLE 8 Show that the following functions are continuous on their natural domains.

(a) $y = \sqrt{x^2 - 2x - 5}$

(b) $y = \frac{x^{2/3}}{1 + x^4}$

(c) $y = \left| \frac{x - 2}{x^2 - 2} \right|$

(d) $y = \left| \frac{x \sin x}{x^2 + 2} \right|$

Solution

- (a) The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $f(x) = x$ (Part 7, Theorem 8). The given function is then the composition of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its natural domain.
- (b) The numerator is the cube root of the identity function squared; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.
- (c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function (Example 7).

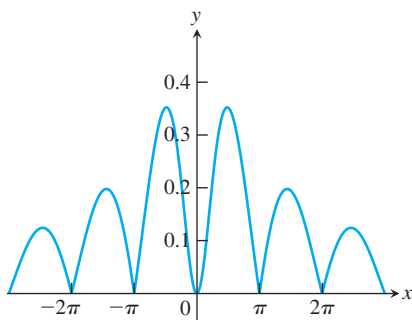


FIGURE 2.65 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous (Example 8d).

(d) Because the sine function is everywhere-continuous (Exercise 76), the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composition of a quotient of continuous functions with the continuous absolute value function (Figure 2.65). ■

Theorem 9 is actually a consequence of a more general result, which we now prove. It states that if the limit of $f(x)$ as x approaches c is equal to b , then the limit of the composition function $g \circ f$ as x approaches c is equal to $g(b)$.

THEOREM 10—Limits of Continuous Functions

If $\lim_{x \rightarrow c} f(x) = b$ and g is continuous at the point b , then

$$\lim_{x \rightarrow c} g(f(x)) = g(b).$$

Proof Let $\varepsilon > 0$ be given. Since g is continuous at b , there exists a number $\delta_1 > 0$ such that

$$|g(y) - g(b)| < \varepsilon \quad \text{whenever} \quad 0 < |y - b| < \delta_1. \quad \begin{array}{l} \text{lim } g(y) = g(b) \text{ since } g \text{ is} \\ \text{continuous at } y = b. \end{array}$$

Note that if $|y - b| = 0$, so that $y = b$, then the inequality $|g(y) - g(b)| < \varepsilon$ holds for any positive ε , and therefore we have

$$|g(y) - g(b)| < \varepsilon \quad \text{whenever} \quad |y - b| < \delta_1. \quad (1)$$

Since $\lim_{x \rightarrow c} f(x) = b$, there exists a $\delta > 0$ such that

$$|f(x) - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta. \quad \begin{array}{l} \text{Definition of } \lim_{x \rightarrow c} f(x) = b \end{array}$$

If we let $y = f(x)$, we then have that

$$|y - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta,$$

which implies from Equation (1) that $|g(y) - g(b)| = |g(f(x)) - g(b)| < \varepsilon$ whenever $0 < |x - c| < \delta$. From the definition of limit, it follows that $\lim_{x \rightarrow c} g(f(x)) = g(b)$. This gives the proof for the case where c is an interior point of the domain of f . The case where c is an endpoint of the domain is entirely similar, using an appropriate one-sided limit in place of a two-sided limit. ■

EXAMPLE 9 As an application of Theorem 10, we have the following calculations.

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow \pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) &= \cos\left(\lim_{x \rightarrow \pi/2} 2x + \lim_{x \rightarrow \pi/2} \sin\left(\frac{3\pi}{2} + x\right)\right) \\ &= \cos(\pi + \sin 2\pi) = \cos \pi = -1. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 1} \sin^{-1}\left(\frac{1-x}{1-x^2}\right) &= \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1-x}{1-x^2}\right) && \text{Arcsine is continuous.} \\ &= \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1}{1+x}\right) && \text{Cancel common factor } (1-x). \\ &= \sin^{-1}\frac{1}{2} = \frac{\pi}{6}. \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \lim_{x \rightarrow 0} \sqrt{x+1} e^{\tan x} &= \lim_{x \rightarrow 0} \sqrt{x+1} \cdot \exp\left(\lim_{x \rightarrow 0} \tan x\right) && \text{exp is continuous.} \\ &= 1 \cdot e^0 = 1. \end{aligned}$$

We sometimes denote e^u by $\exp(u)$ when u is a complicated mathematical expression.

Intermediate Value Theorem for Continuous Functions

A function is said to have the **Intermediate Value Property** if whenever it takes on two values, it also takes on all the values in between.

THEOREM 11 – The Intermediate Value Theorem for Continuous Functions

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

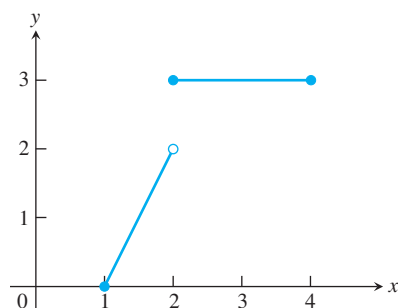
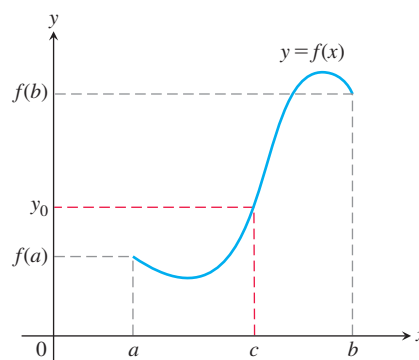


FIGURE 2.66 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \leq x < 2 \\ 3, & 2 \leq x \leq 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

Theorem 11 says that continuous functions over *finite closed* intervals have the Intermediate Value Property. Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the y -axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.

The proof of the Intermediate Value Theorem depends on the completeness property of the real number system. The completeness property implies that the real numbers have no holes or gaps. In contrast, the rational numbers do not satisfy the completeness property, and a function defined only on the rationals would not satisfy the Intermediate Value Theorem. See Appendix A.9 for a discussion and examples.

The continuity of f on the interval is essential to Theorem 11. If f fails to be continuous at even one point of the interval, the theorem's conclusion may fail, as it does for the function graphed in Figure 2.66 (choose y_0 as any number between 2 and 3).

A Consequence for Graphing: Connectedness Theorem 11 implies that the graph of a function that is continuous on an interval cannot have any breaks over the interval. It will be **connected**—a single, unbroken curve. It will not have jumps such as the ones found in the graph of the greatest integer function (Figure 2.61), or separate branches as found in the graph of $1/x$ (Figure 2.63).

A Consequence for Root Finding We call a solution of the equation $f(x) = 0$ a **root** of the equation or a **zero** of the function f . The Intermediate Value Theorem tells us that if f is continuous, then any interval on which f changes sign contains a zero of the function. Somewhere between a point where a continuous function is positive and a second point where it is negative, the function must be equal to zero.

In practical terms, when we see the graph of a continuous function cross the horizontal axis on a computer screen, we know it is not stepping across. There really is a point where the function's value is zero.

EXAMPLE 10 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Solution Let $f(x) = x^3 - x - 1$. Since $f(1) = 1 - 1 - 1 = -1 < 0$ and $f(2) = 2^3 - 2 - 1 = 5 > 0$, we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$. Since f is a polynomial, it is continuous, and the Intermediate Value Theorem says there is a zero of f between 1 and 2. Figure 2.67 shows the result of zooming in to locate a root near $x = 1.32$. ■

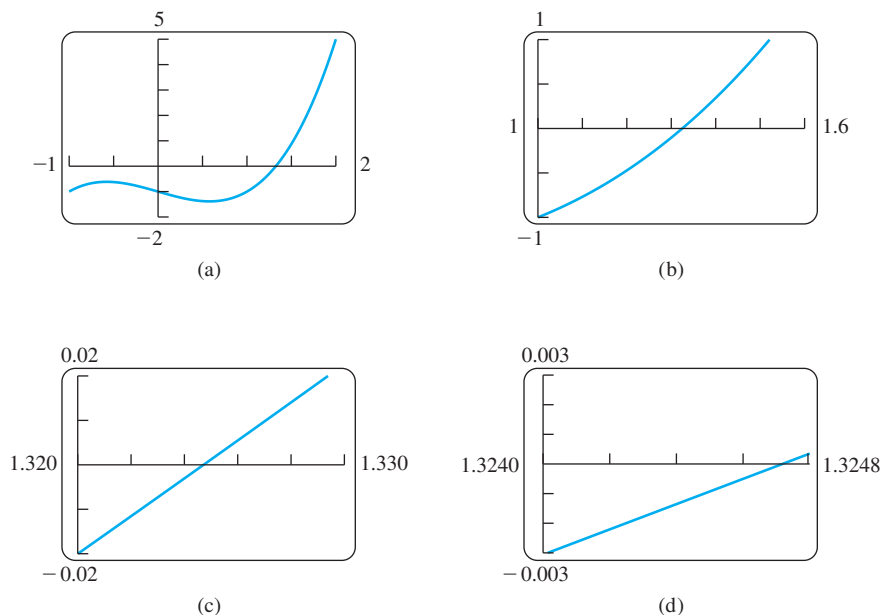


FIGURE 2.67 Zooming in on a zero of the function $f(x) = x^3 - x - 1$. The zero is near $x = 1.3247$ (Example 10).

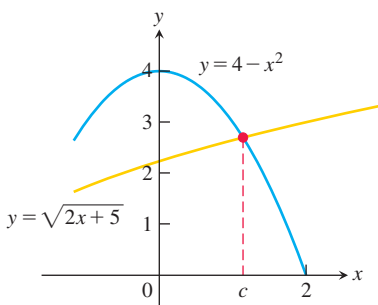


FIGURE 2.68 The curves $y = \sqrt{2x + 5}$ and $y = 4 - x^2$ have the same value at the number $x = c$ where $\sqrt{2x + 5} + x^2 - 4 = 0$ (Example 11).

EXAMPLE 11 Use the Intermediate Value Theorem to prove that the equation

$$\sqrt{2x + 5} = 4 - x^2$$

has a solution (Figure 2.68).

Solution We rewrite the equation as

$$\sqrt{2x + 5} + x^2 - 4 = 0,$$

and set $f(x) = \sqrt{2x + 5} + x^2 - 4$. Now $g(x) = \sqrt{2x + 5}$ is continuous on the interval $[-5/2, \infty)$ since it is formed as the composition of two continuous functions, the square root function with the nonnegative linear function $y = 2x + 5$. Then f is the sum of the function g and the quadratic function $y = x^2 - 4$, and the quadratic function is continuous for all values of x . It follows that $f(x) = \sqrt{2x + 5} + x^2 - 4$ is continuous on the interval $[-5/2, \infty)$. By trial and error, we find the function values $f(0) = \sqrt{5} - 4 \approx -1.76$ and $f(2) = \sqrt{9} = 3$. Note that f is continuous on the finite closed interval $[0, 2]$, which is a subset of the domain $[-5/2, \infty)$. Since the value $y_0 = 0$ is between the numbers $f(0) = -1.76$ and $f(2) = 3$, by the Intermediate Value Theorem there is a number $c \in [0, 2]$ such that $f(c) = 0$. We have found a number c that solves the original equation. ■

Continuous Extension to a Point

Sometimes the formula that describes a function f does not make sense at a point $x = c$. It might nevertheless be possible to extend the domain of f to include $x = c$, creating a new function that is continuous at $x = c$. For example, the function $y = f(x) = (\sin x)/x$ is continuous at every point except $x = 0$, since $x = 0$ is not in its domain. Since $y = (\sin x)/x$ has a finite limit as $x \rightarrow 0$ (Theorem 6), we can extend the function's domain to include the point $x = 0$ in such a way that the extended function is continuous at $x = 0$. We define the new function

$$F(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \quad \text{Same as original function for } x \neq 0 \\ 1, & x = 0. \quad \text{Value at domain point } x = 0 \end{cases}$$

The new function $F(x)$ is continuous at $x = 0$ because

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = F(0),$$

so it meets the requirements for continuity (Figure 2.69).

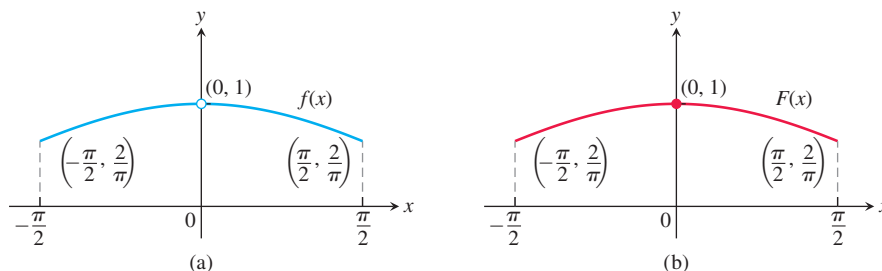


FIGURE 2.69 (a) The graph of $f(x) = (\sin x)/x$ for $-\pi/2 \leq x \leq \pi/2$ does not include the point $(0, 1)$ because the function is not defined at $x = 0$. (b) We can extend the domain to include $x = 0$ by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \rightarrow 0} f(x)$ and $F(x)$ is a continuous function at $x = 0$.

More generally, a function (such as a rational function) may have a limit at a point where it is not defined. If $f(c)$ is not defined, but $\lim_{x \rightarrow c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) = \begin{cases} f(x), & \text{if } x \text{ is in the domain of } f \\ L, & \text{if } x = c. \end{cases}$$

The function F is continuous at $x = c$. It is called the **continuous extension of f** to $x = c$. For rational functions f , continuous extensions are often found by canceling common factors in the numerator and denominator.

EXAMPLE 12 Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}.$$

The new function

$$F(x) = \frac{x+3}{x+2}$$

is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \rightarrow 2} f(x) = \frac{5}{4}.$$

The graph of f is shown in Figure 2.70. The continuous extension F has the same graph except with no hole at $(2, 5/4)$. Effectively, F is the function f extended across the missing domain point at $x = 2$ so as to give a continuous function over the larger domain. ■

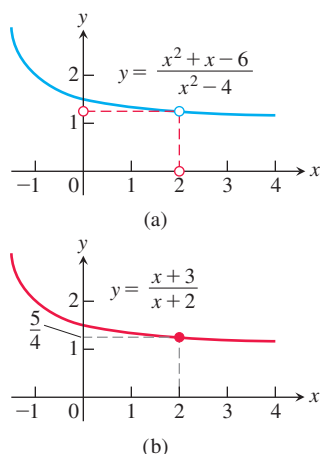
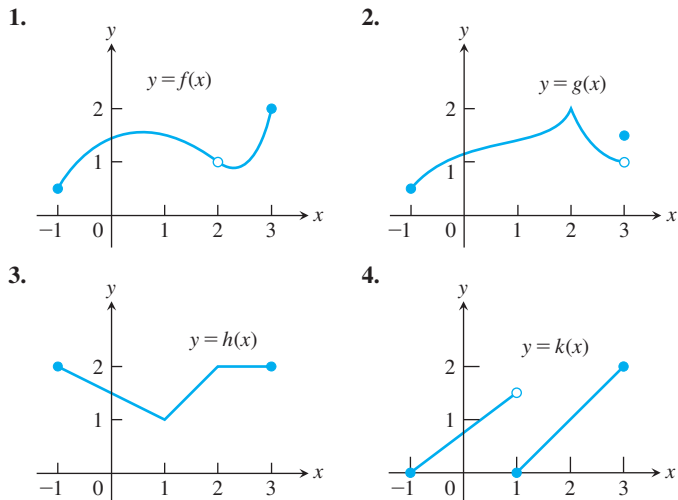


FIGURE 2.70 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 12).

EXERCISES 2.6

Continuity from Graphs

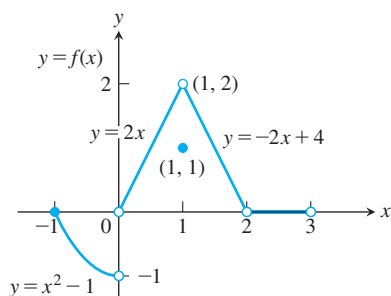
In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?



Exercises 5–10 refer to the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \leq x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.



5. a. Does $f(-1)$ exist?
b. Does $\lim_{x \rightarrow -1^+} f(x)$ exist?
c. Does $\lim_{x \rightarrow -1^+} f(x) = f(-1)$?
d. Is f continuous at $x = -1$?
6. a. Does $f(1)$ exist?
b. Does $\lim_{x \rightarrow 1} f(x)$ exist?
c. Does $\lim_{x \rightarrow 1} f(x) = f(1)$?
d. Is f continuous at $x = 1$?
7. a. Is f defined at $x = 2$? (Look at the definition of f).
b. Is f continuous at $x = 2$?
8. At what values of x is f continuous?

9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?
10. To what new value should $f(1)$ be changed to remove the discontinuity?

Applying the Continuity Test

At which points do the functions in Exercises 11 and 12 fail to be continuous? At which points, if any, are the discontinuities removable? Not removable? Give reasons for your answers.

11. The function defined in Exercise 1, Section 2.4
12. The function defined in Exercise 2, Section 2.4

At what points are the functions in Exercises 13–32 continuous?

13. $y = \frac{1}{x-2} - 3x$
14. $y = \frac{1}{(x+2)^2} + 4$
15. $y = \frac{x+1}{x^2-4x+3}$
16. $y = \frac{x+3}{x^2-3x-10}$
17. $y = |x-1| + \sin x$
18. $y = \frac{1}{|x|+1} - \frac{x^2}{2}$
19. $y = \frac{\cos x}{x}$
20. $y = \frac{x+2}{\cos x}$
21. $y = \csc 2x$
22. $y = \tan \frac{\pi x}{2}$
23. $y = \frac{x \tan x}{x^2+1}$
24. $y = \frac{\sqrt{x^4+1}}{1+\sin^2 x}$
25. $y = \sqrt{2x+3}$
26. $y = \sqrt[4]{3x-1}$
27. $y = (2x-1)^{1/3}$
28. $y = (2-x)^{1/5}$

29. $g(x) = \begin{cases} \frac{x^2-x-6}{x-3}, & x \neq 3 \\ 5, & x = 3 \end{cases}$
30. $f(x) = \begin{cases} \frac{x^3-8}{x^2-4}, & x \neq 2, x \neq -2 \\ 3, & x = 2 \\ 4, & x = -2 \end{cases}$
31. $f(x) = \begin{cases} 1-x, & x < 0 \\ e^x, & 0 \leq x \leq 1 \\ x^2+2, & x > 1 \end{cases}$
32. $f(x) = \frac{x+3}{2-e^x}$

Limits Involving Trigonometric Functions

Find the limits in Exercises 33–40. Are the functions continuous at the point being approached?

33. $\lim_{x \rightarrow \pi} \sin(x - \sin x)$
34. $\lim_{t \rightarrow 0} \sin\left(\frac{\pi}{2} \cos(\tan t)\right)$
35. $\lim_{y \rightarrow 1} \sec(y \sec^2 y - \tan^2 y - 1)$
36. $\lim_{x \rightarrow 0} \tan\left(\frac{\pi}{4} \cos(\sin x^{1/3})\right)$
37. $\lim_{t \rightarrow 0} \cos\left(\frac{\pi}{\sqrt{19-3 \sec 2t}}\right)$
38. $\lim_{x \rightarrow \pi/6} \sqrt{\csc^2 x + 5\sqrt{3} \tan x}$

$$39. \lim_{x \rightarrow 0^+} \sin\left(\frac{\pi}{2}e^{\sqrt{x}}\right) \qquad 40. \lim_{x \rightarrow 1} \cos^{-1}(\ln \sqrt{x})$$

$$41. \lim_{x \rightarrow 0} \sec\left[e^x + \pi \tan\left(\frac{\pi}{4 \sec x}\right) - 1\right]$$

$$42. \lim_{x \rightarrow 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2 \sec x}\right)$$

$$43. \lim_{t \rightarrow 0} \tan\left(1 - \frac{\sin t}{t}\right) \qquad 44. \lim_{\theta \rightarrow 0} \cos\left(\frac{\pi\theta}{\sin \theta}\right)$$

Continuous Extensions

45. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.

46. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

47. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.

48. Define $g(4)$ in a way that extends

$$g(x) = (x^2 - 16)/(x^2 - 3x - 4)$$

to be continuous at $x = 4$.

49. For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x ?

50. For what value of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x ?

51. For what values of a is

$$f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$$

continuous at every x ?

52. For what values of b is

$$g(x) = \begin{cases} \frac{x-b}{b+1}, & x \leq 0 \\ x^2 + b, & x > 0 \end{cases}$$

continuous at every x ?

53. For what values of a and b is

$$f(x) = \begin{cases} -2, & x \leq -1 \\ ax - b, & -1 < x < 1 \\ 3, & x \geq 1 \end{cases}$$

continuous at every x ?

54. For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x ?

T In Exercises 55–58, graph the function f to see whether it appears to have a continuous extension to $x = 0$. If it does, use Trace and Zoom to find a good candidate for the extended function's value at $x = 0$. If the function does not appear to have a continuous extension, can it be extended to be continuous at $x = 0$ from the right or from the left? If so, what do you think the extended function's value(s) should be?

$$55. f(x) = \frac{10^x - 1}{x}$$

$$56. f(x) = \frac{10^{|x|} - 1}{x}$$

$$57. f(x) = \frac{\sin x}{|x|}$$

$$58. f(x) = (1 + 2x)^{1/x}$$

Theory and Examples

59. A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

60. Explain why the equation $\cos x = x$ has at least one solution.

61. **Roots of a cubic** Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

62. **A function value** Show that the function $F(x) = (x - a)^2 \cdot (x - b)^2 + x$ takes on the value $(a + b)/2$ for some value of x .

63. **Solving an equation** If $f(x) = x^3 - 8x + 10$, show that there are values c for which $f(c)$ equals (a) π ; (b) $-\sqrt{3}$; (c) 5,000,000.

64. Explain why the following five statements ask for the same information.

a. Find the roots of $f(x) = x^3 - 3x - 1$.

b. Find the x -coordinates of the points where the curve $y = x^3$ crosses the line $y = 3x + 1$.

c. Find all the values of x for which $x^3 - 3x = 1$.

d. Find the x -coordinates of the points where the cubic curve $y = x^3 - 3x$ crosses the line $y = 1$.

e. Solve the equation $x^3 - 3x - 1 = 0$.

65. **Removable discontinuity** Give an example of a function $f(x)$ that is continuous for all values of x except $x = 2$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 2$, and how you know the discontinuity is removable.

66. **Nonremovable discontinuity** Give an example of a function $g(x)$ that is continuous for all values of x except $x = -1$, where it has a nonremovable discontinuity. Explain how you know that g is discontinuous there and why the discontinuity is not removable.

67. **A function discontinuous at every point**

a. Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

b. Is f right-continuous or left-continuous at any point?

68. If functions $f(x)$ and $g(x)$ are continuous for $0 \leq x \leq 1$, could $f(x)/g(x)$ possibly be discontinuous at a point of $[0, 1]$? Give reasons for your answer.

69. If the product function $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, must $f(x)$ and $g(x)$ be continuous at $x = 0$? Give reasons for your answer.

- 70. Discontinuous compositions of continuous functions** Give an example of functions f and g , both continuous at $x = 0$, for which the composition $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 9? Give reasons for your answer.
- 71. Never-zero continuous functions** Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.
- 72. Stretching a rubber band** Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? That is, if x is a position on the rubber band before stretching and $f(x)$ is the position of that point after stretching, must there be some x such that $f(x) = x$? Give reasons for your answer.
- 73. A fixed point theorem** Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a **fixed point** of f).
- 74. The sign-preserving property of continuous functions** Let f be defined on an interval (a, b) and suppose that $f(c) \neq 0$ at some c where f is continuous. Show that there is an interval $(c - \delta, c + \delta)$ about c where f has the same sign as $f(c)$.

- 75.** Prove that f is continuous at c if and only if

$$\lim_{h \rightarrow 0} f(c + h) = f(c).$$

- 76.** Use Exercise 75 together with the identities

$$\sin(h + c) = \sin h \cos c + \cos h \sin c,$$

$$\cos(h + c) = \cos h \cos c - \sin h \sin c$$

to prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

Solving Equations Graphically

T Use the Intermediate Value Theorem in Exercises 77–84 to prove that each equation has a solution. Then use a graphing calculator or computer grapher to solve the equations.

77. $x^3 - 3x - 1 = 0$

78. $2x^3 - 2x^2 - 2x + 1 = 0$

79. $x(x - 1)^2 = 1$ (one root) **80.** $x^x = 2$

81. $\sqrt{x} + \sqrt{1 + x} = 4$

82. $x^3 - 15x + 1 = 0$ (three roots)

83. $\cos x = x$ (one root). Make sure you are using radian mode.

84. $2 \sin x = x$ (three roots). Make sure you are using radian mode.

CHAPTER 2 Questions to Guide Your Review

- What is the average rate of change of the function $g(t)$ over the interval from $t = a$ to $t = b$? How is it related to a secant line?
- What limit must be calculated to find the rate of change of a function $g(t)$ at $t = t_0$?
- Give an informal or intuitive definition of the limit

$$\lim_{x \rightarrow c} f(x) = L.$$

Why is the definition “informal”? Give examples.

- Does the existence and value of the limit of a function $f(x)$ as x approaches c ever depend on what happens at $x = c$? Explain and give examples.
- What function behaviors might occur for which the limit may fail to exist? Give examples.
- What theorems are available for calculating limits? Give examples of how the theorems are used.
- How are one-sided limits related to limits? How can this relationship sometimes be used to calculate a limit or prove it does not exist? Give examples.
- What is the value of $\lim_{\theta \rightarrow 0} ((\sin \theta)/\theta)$? Does it matter whether θ is measured in degrees or radians? Explain.
- What exactly does $\lim_{x \rightarrow c} f(x) = L$ mean? Give an example in which you find $\delta > 0$ for a given f, L, c , and $\varepsilon > 0$ in the precise definition of limit.
- Give precise definitions of the following statements.
 - $\lim_{x \rightarrow 2^-} f(x) = 5$
 - $\lim_{x \rightarrow 2^+} f(x) = 5$
 - $\lim_{x \rightarrow 2} f(x) = \infty$
 - $\lim_{x \rightarrow 2} f(x) = -\infty$

- What conditions must be satisfied by a function if it is to be continuous at an interior point of its domain? At an endpoint?
- How can looking at the graph of a function help you tell where the function is continuous?
- What does it mean for a function to be right-continuous at a point? Left-continuous? How are continuity and one-sided continuity related?
- What does it mean for a function to be continuous on an interval? Give examples to illustrate the fact that a function that is not continuous on its entire domain may still be continuous on selected intervals within the domain.
- What are the basic types of discontinuity? Give an example of each. What is a removable discontinuity? Give an example.
- What does it mean for a function to have the Intermediate Value Property? What conditions guarantee that a function has this property over an interval? What are the consequences for graphing and solving the equation $f(x) = 0$?
- Under what circumstances can you extend a function $f(x)$ to be continuous at a point $x = c$? Give an example.
- What exactly do $\lim_{x \rightarrow \infty} f(x) = L$ and $\lim_{x \rightarrow -\infty} f(x) = L$ mean? Give examples.
- What are $\lim_{x \rightarrow \pm\infty} k$ (k a constant) and $\lim_{x \rightarrow \pm\infty} (1/x)$? How do you extend these results to other functions? Give examples.
- How do you find the limit of a rational function as $x \rightarrow \pm\infty$? Give examples.
- What are horizontal and vertical asymptotes? Give examples.

CHAPTER 2 Practice Exercises

Limits and Continuity

1. Graph the function

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1. \end{cases}$$

Then discuss, in detail, limits, one-sided limits, continuity, and one-sided continuity of f at $x = -1, 0$, and 1 . Are any of the discontinuities removable? Explain.

2. Repeat the instructions of Exercise 1 for

$$f(x) = \begin{cases} 0, & x \leq -1 \\ 1/x, & 0 < |x| < 1 \\ 0, & x = 1 \\ 1, & x > 1. \end{cases}$$

3. Suppose that $f(t)$ and $g(t)$ are defined for all t and that $\lim_{t \rightarrow t_0} f(t) = -7$ and $\lim_{t \rightarrow t_0} g(t) = 0$. Find the limit as $t \rightarrow t_0$ of the following functions.

- | | |
|----------------------|----------------------------|
| a. $3f(t)$ | b. $(f(t))^2$ |
| c. $f(t) \cdot g(t)$ | d. $\frac{f(t)}{g(t) - 7}$ |
| e. $\cos(g(t))$ | f. $ f(t) $ |
| g. $f(t) + g(t)$ | h. $1/f(t)$ |

4. Suppose the functions $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x \rightarrow 0} f(x) = 1/2$ and $\lim_{x \rightarrow 0} g(x) = \sqrt{2}$. Find the limits as $x \rightarrow 0$ of the following functions.

- | | |
|------------------|--------------------------------------|
| a. $-g(x)$ | b. $g(x) \cdot f(x)$ |
| c. $f(x) + g(x)$ | d. $1/f(x)$ |
| e. $x + f(x)$ | f. $\frac{f(x) \cdot \cos x}{x - 1}$ |

In Exercises 5 and 6, find the value that $\lim_{x \rightarrow 0} g(x)$ must have if the given limit statements hold.

5. $\lim_{x \rightarrow 0} \left(\frac{4 - g(x)}{x} \right) = 1$ 6. $\lim_{x \rightarrow -4} \left(x \lim_{x \rightarrow 0} g(x) \right) = 2$

7. On what intervals are the following functions continuous?

- | | |
|----------------------|----------------------|
| a. $f(x) = x^{1/3}$ | b. $g(x) = x^{3/4}$ |
| c. $h(x) = x^{-2/3}$ | d. $k(x) = x^{-1/6}$ |

8. On what intervals are the following functions continuous?

- | | |
|------------------------------------|------------------------------|
| a. $f(x) = \tan x$ | b. $g(x) = \csc x$ |
| c. $h(x) = \frac{\cos x}{x - \pi}$ | d. $k(x) = \frac{\sin x}{x}$ |

Finding Limits

In Exercises 9–28, find the limit or explain why it does not exist.

9. $\lim_{x \rightarrow 0} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$
 a. as $x \rightarrow 0$ b. as $x \rightarrow 2$

10. $\lim_{x \rightarrow 1} \frac{x^2 + x}{x^5 + 2x^4 + x^3}$

- a. as $x \rightarrow 0$

- b. as $x \rightarrow -1$

11. $\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}$

12. $\lim_{x \rightarrow a} \frac{x^2 - a^2}{x^4 - a^4}$

13. $\lim_{h \rightarrow 0} \frac{(x + h)^2 - x^2}{h}$

14. $\lim_{x \rightarrow 0} \frac{(x + h)^2 - x^2}{h}$

15. $\lim_{x \rightarrow 0} \frac{1}{\frac{2}{2+x} - \frac{1}{2}}$

17. $\lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{\sqrt{x} - 1}$

19. $\lim_{x \rightarrow 0} \frac{\tan(2x)}{\tan(\pi x)}$

21. $\lim_{x \rightarrow \pi} \sin\left(\frac{x}{2} + \sin x\right)$

23. $\lim_{x \rightarrow 0} \frac{8x}{3 \sin x - x}$

25. $\lim_{t \rightarrow 3^+} \ln(t - 3)$

27. $\lim_{\theta \rightarrow 0^+} \sqrt{\theta} e^{\cos(\pi/\theta)}$

16. $\lim_{x \rightarrow 0} \frac{(2 + x)^3 - 8}{x}$

18. $\lim_{x \rightarrow 64} \frac{x^{2/3} - 16}{\sqrt{x} - 8}$

20. $\lim_{x \rightarrow \pi^-} \csc x$

22. $\lim_{x \rightarrow \pi} \cos^2(x - \tan x)$

24. $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\sin x}$

26. $\lim_{t \rightarrow 1} t^2 \ln(2 - \sqrt{t})$

28. $\lim_{z \rightarrow 0^+} \frac{2e^{1/z}}{e^{1/z} + 1}$

In Exercises 29–32, find the limit of $g(x)$ as x approaches the indicated value.

29. $\lim_{x \rightarrow 0^+} (4g(x))^{1/3} = 2$

30. $\lim_{x \rightarrow \sqrt{5}} \frac{1}{x + g(x)} = 2$

31. $\lim_{x \rightarrow 1} \frac{3x^2 + 1}{g(x)} = \infty$

32. $\lim_{x \rightarrow -2} \frac{5 - x^2}{\sqrt{g(x)}} = 0$

T Roots

33. Let $f(x) = x^3 - x - 1$.

- Use the Intermediate Value Theorem to show that f has a zero between -1 and 2 .
- Solve the equation $f(x) = 0$ graphically with an error of magnitude at most 10^{-8} .
- It can be shown that the exact value of the solution in part (b) is

$$\left(\frac{1}{2} + \frac{\sqrt{69}}{18} \right)^{1/3} + \left(\frac{1}{2} - \frac{\sqrt{69}}{18} \right)^{1/3}.$$

Evaluate this exact answer and compare it with the value you found in part (b).

34. Let $f(\theta) = \theta^3 - 2\theta + 2$.

- Use the Intermediate Value Theorem to show that f has a zero between -2 and 0 .

- b. Solve the equation $f(\theta) = 0$ graphically with an error of magnitude at most 10^{-4} .
- c. It can be shown that the exact value of the solution in part (b) is

$$\left(\sqrt{\frac{19}{27}} - 1\right)^{1/3} - \left(\sqrt{\frac{19}{27}} + 1\right)^{1/3}.$$

Evaluate this exact answer and compare it with the value you found in part (b).

Continuous Extension

35. Can $f(x) = x(x^2 - 1)/|x^2 - 1|$ be extended to be continuous at $x = 1$ or -1 ? Give reasons for your answers. (Graph the function—you will find the graph interesting.)
36. Explain why the function $f(x) = \sin(1/x)$ has no continuous extension to $x = 0$.

T In Exercises 37–40, graph the function to see whether it appears to have a continuous extension to the given point a . If it does, use Trace and Zoom to find a good candidate for the extended function's value at a . If the function does not appear to have a continuous extension, can it be extended to be continuous from the right or left? If so, what do you think the extended function's value should be?

37. $f(x) = \frac{x-1}{x-\sqrt[4]{x}}$, $a = 1$

38. $g(\theta) = \frac{5 \cos \theta}{4\theta - 2\pi}$, $a = \pi/2$

39. $h(t) = (1 + |t|)^{1/t}$, $a = 0$

40. $k(x) = \frac{x}{1 - 2^{|x|}}$, $a = 0$

Limits at Infinity

Find the limits in Exercises 41–54.

41. $\lim_{x \rightarrow \infty} \frac{2x+3}{5x+7}$

42. $\lim_{x \rightarrow -\infty} \frac{2x^2+3}{5x^2+7}$

43. $\lim_{x \rightarrow -\infty} \frac{x^2-4x+8}{3x^3}$

44. $\lim_{x \rightarrow \infty} \frac{1}{x^2-7x+1}$

45. $\lim_{x \rightarrow -\infty} \frac{x^2-7x}{x+1}$

46. $\lim_{x \rightarrow \infty} \frac{x^4+x^3}{12x^3+128}$

47. $\lim_{x \rightarrow \infty} \frac{\sin x}{|x|}$ (If you have a grapher, try graphing the function for $-5 \leq x \leq 5$.)

48. $\lim_{\theta \rightarrow \infty} \frac{\cos \theta - 1}{\theta}$ (If you have a grapher, try graphing $f(x) = x(\cos(1/x) - 1)$ near the origin to “see” the limit at infinity.)

49. $\lim_{x \rightarrow \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

50. $\lim_{x \rightarrow \infty} \frac{x^{2/3} + x^{-1}}{x^{2/3} + \cos^2 x}$

51. $\lim_{x \rightarrow \infty} e^{1/x} \cos \frac{1}{x}$

52. $\lim_{t \rightarrow \infty} \ln\left(1 + \frac{1}{t}\right)$

53. $\lim_{x \rightarrow -\infty} \tan^{-1} x$

54. $\lim_{t \rightarrow -\infty} e^{3t} \sin^{-1}\left(\frac{1}{t}\right)$

Horizontal and Vertical Asymptotes

55. Use limits to determine the equations for all vertical asymptotes.

a. $y = \frac{x^2+4}{x-3}$

b. $f(x) = \frac{x^2-x-2}{x^2-2x+1}$

c. $y = \frac{x^2+x-6}{x^2+2x-8}$

56. Use limits to determine the equations for all horizontal asymptotes.

a. $y = \frac{1-x^2}{x^2+1}$

b. $f(x) = \frac{\sqrt{x}+4}{\sqrt{x}+4}$

c. $g(x) = \frac{\sqrt{x^2+4}}{x}$

d. $y = \sqrt{\frac{x^2+9}{9x^2+1}}$

57. Determine the domain and range of $y = \frac{\sqrt{16-x^2}}{x-2}$.

58. Assume that constants a and b are positive. Find equations for all horizontal and vertical asymptotes for the graph of $y = \frac{\sqrt{ax^2+4}}{x-b}$.

CHAPTER 2 Additional and Advanced Exercises

- T** 1. **Assigning a value to 0^0** The rules of exponents tell us that $a^0 = 1$ if a is any number different from zero. They also tell us that $0^n = 0$ if n is any positive number.

If we tried to extend these rules to include the case 0^0 , we would get conflicting results. The first rule would say $0^0 = 1$, whereas the second would say $0^0 = 0$.

We are not dealing with a question of right or wrong here. Neither rule applies as it stands, so there is no contradiction. We could, in fact, define 0^0 to have any value we wanted as long as we could persuade others to agree.

What value would you like 0^0 to have? Here is an example that might help you to decide. (See Exercise 2 below for another example.)

- a. Calculate x^x for $x = 0.1, 0.01, 0.001$, and so on as far as your calculator can go. Record the values you get. What pattern do you see?

- b. Graph the function $y = x^x$ for $0 < x \leq 1$. Even though the function is not defined for $x \leq 0$, the graph will approach the y -axis from the right. Toward what y -value does it seem to be headed? Zoom in to further support your idea.

- T** 2. **A reason you might want 0^0 to be something other than 0 or 1** As the number x increases through positive values, the numbers $1/x$ and $1/(\ln x)$ both approach zero. What happens to the number

$$f(x) = \left(\frac{1}{x}\right)^{1/(\ln x)}$$

as x increases? Here are two ways to find out.

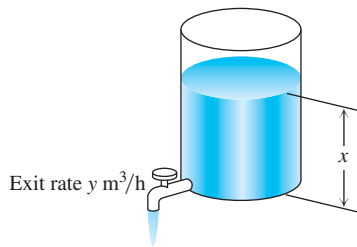
- a. Evaluate f for $x = 10, 100, 1000$, and so on as far as your calculator can reasonably go. What pattern do you see?
- b. Graph f in a variety of graphing windows, including windows that contain the origin. What do you see? Trace the y -values along the graph. What do you find?

- 3. Lorentz contraction** In relativity theory, the length of an object, say a rocket, appears to an observer to depend on the speed at which the object is traveling with respect to the observer. If the observer measures the rocket's length as L_0 at rest, then at speed v the length will appear to be

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}.$$

This equation is the Lorentz contraction formula. Here, c is the speed of light in a vacuum, about 3×10^8 m/s. What happens to L as v increases? Find $\lim_{v \rightarrow c^-} L$. Why was the left-hand limit needed?

- 4. Controlling the flow from a draining tank** Torricelli's law says that if you drain a tank like the one in the figure shown, the rate y at which water runs out is a constant times the square root of the water's depth x . The constant depends on the size and shape of the exit valve.



Suppose that $y = \sqrt{x}/2$ for a certain tank. You are trying to maintain a fairly constant exit rate by adding water to the tank with a hose from time to time. How deep must you keep the water if you want to maintain the exit rate

- within $0.2 \text{ m}^3/\text{h}$ of the rate $y_0 = 1 \text{ m}^3/\text{h}$?
- within $0.1 \text{ m}^3/\text{h}$ of the rate $y_0 = 1 \text{ m}^3/\text{h}$?

- 5. Thermal expansion in precise equipment** As you may know, most metals expand when heated and contract when cooled. The dimensions of a piece of laboratory equipment are sometimes so critical that the shop where the equipment is made must be held at the same temperature as the laboratory where the equipment is to be used. A typical aluminum bar that is 10 cm wide at 20°C will be

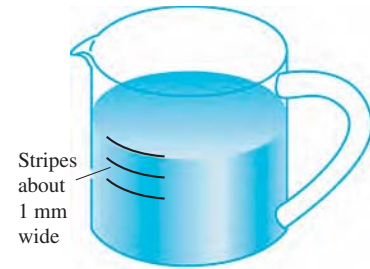
$$y = 10 + 2(t - 20) \times 10^{-4}$$

Centimeters wide at a nearby temperature t . Suppose that you are using a bar like this in a gravity wave detector, where its width must stay within 0.0005 cm of the ideal 10 cm. How close to $t_0 = 20^\circ\text{C}$ must you maintain the temperature to ensure that this tolerance is not exceeded?

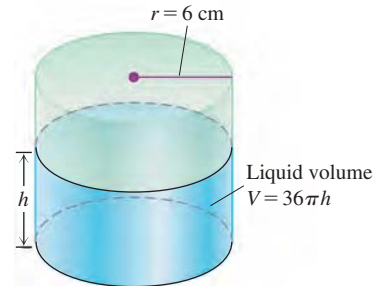
- 6. Stripes on a measuring cup** The interior of a typical 1-L measuring cup is a right circular cylinder of radius 6 cm (see accompanying figure). The volume of water we put in the cup is therefore a function of the level h to which the cup is filled, the formula being

$$V = \pi 6^2 h = 36\pi h.$$

How closely must we measure h to measure out 1 L of water (1000 cm^3) with an error of no more than 1% (10 cm^3)?



(a)



(b)

A 1-L measuring cup (a), modeled as a right circular cylinder (b) of radius $r = 6 \text{ cm}$

Precise Definition of Limit

In Exercises 7–10, use the formal definition of limit to prove that the function is continuous at c .

7. $f(x) = x^2 - 7$, $c = 1$ 8. $g(x) = 1/(2x)$, $c = 1/4$

9. $h(x) = \sqrt{2x - 3}$, $c = 2$ 10. $F(x) = \sqrt{9 - x}$, $c = 5$

- 11. Uniqueness of limits** Show that a function cannot have two different limits at the same point. That is, if $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} f(x) = L_2$, then $L_1 = L_2$.

- 12. Prove the limit Constant Multiple Rule:**

$$\lim_{x \rightarrow c} kf(x) = k \lim_{x \rightarrow c} f(x) \quad \text{for any constant } k.$$

- 13. One-sided limits** If $\lim_{x \rightarrow 0^+} f(x) = A$ and $\lim_{x \rightarrow 0^-} f(x) = B$, find

- $\lim_{x \rightarrow 0^+} f(x^3 - x)$
- $\lim_{x \rightarrow 0^-} f(x^3 - x)$
- $\lim_{x \rightarrow 0^+} f(x^2 - x^4)$
- $\lim_{x \rightarrow 0^-} f(x^2 - x^4)$

- 14. Limits and continuity** Which of the following statements are true, and which are false? If true, say why; if false, give a counterexample (that is, an example confirming the falsehood).

- a. If $\lim_{x \rightarrow c} f(x)$ exists but $\lim_{x \rightarrow c} g(x)$ does not exist, then

$$\lim_{x \rightarrow c} (f(x) + g(x)) \text{ does not exist.}$$

- b. If neither $\lim_{x \rightarrow c} f(x)$ nor $\lim_{x \rightarrow c} g(x)$ exists, then

$$\lim_{x \rightarrow c} (f(x) + g(x)) \text{ does not exist.}$$

- c. If f is continuous at x , then so is $|f|$.
 d. If $|f|$ is continuous at c , then so is f .

In Exercises 15 and 16, use the formal definition of limit to prove that the function has a continuous extension to the given value of x .

15. $f(x) = \frac{x^2 - 1}{x + 1}, \quad x = -1$

16. $g(x) = \frac{x^2 - 2x - 3}{2x - 6}, \quad x = 3$

17. **A function continuous at only one point** Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

- a. Show that f is continuous at $x = 0$.
 b. Use the fact that every nonempty open interval of real numbers contains both rational and irrational numbers to show that f is not continuous at any nonzero value of x .
 18. **The Dirichlet ruler function** If x is a rational number, then x can be written in a unique way as a quotient of integers m/n , where $n > 0$ and m and n have no common factors greater than 1. (We say that such a fraction is in *lowest terms*. For example, $6/4$ written in lowest terms is $3/2$.) Let $f(x)$ be defined for all x in the interval $[0, 1]$ by

$$f(x) = \begin{cases} 1/n & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

For instance, $f(0) = f(1) = 1$, $f(1/2) = 1/2$, $f(1/3) = f(2/3) = 1/3$, $f(1/4) = f(3/4) = 1/4$, and so on.

- a. Show that f is discontinuous at every rational number in $[0, 1]$.
 b. Show that f is continuous at every irrational number in $[0, 1]$.
 (Hint: If ε is a given positive number, show that there are only finitely many rational numbers r in $[0, 1]$ such that $f(r) \geq \varepsilon$.)
 c. Sketch the graph of f . Why do you think f is called the “ruler function”?
 19. **Antipodal points** Is there any reason to believe that there is always a pair of antipodal (diametrically opposite) points on Earth’s equator where the temperatures are the same? Explain.
 20. If $\lim_{x \rightarrow c} (f(x) + g(x)) = 3$ and $\lim_{x \rightarrow c} (f(x) - g(x)) = -1$, find $\lim_{x \rightarrow c} f(x)g(x)$.
 21. **Roots of a quadratic equation that is almost linear** The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \neq 0$, one positive and one negative:

$$r_+(a) = \frac{-1 + \sqrt{1+a}}{a}, \quad r_-(a) = \frac{-1 - \sqrt{1+a}}{a}.$$

- a. What happens to $r_+(a)$ as $a \rightarrow 0$? As $a \rightarrow -1^+$?
 b. What happens to $r_-(a)$ as $a \rightarrow 0$? As $a \rightarrow -1^+$?
 c. Support your conclusions by graphing $r_+(a)$ and $r_-(a)$ as functions of a . Describe what you see.
 d. For added support, graph $f(x) = ax^2 + 2x - 1$ simultaneously for $a = 1, 0.5, 0.2, 0.1$, and 0.05 .
 22. **Root of an equation** Show that the equation $x + 2 \cos x = 0$ has at least one solution.
 23. **Bounded functions** A real-valued function f is **bounded from above** on a set D if there exists a number N such that $f(x) \leq N$ for all x in D . We call N , when it exists, an **upper bound** for f on D and say that f is **bounded from above** by N . In a similar manner, we say that f is **bounded from below** on D if there exists a number M such that $f(x) \geq M$ for all x in D . We call M , when it

exists, a **lower bound** for f on D and say that f is **bounded** from below by M . We say that f is **bounded** on D if it is bounded from both above and below.

- a. Show that f is bounded on D if and only if there exists a number B such that $|f(x)| \leq B$ for all x in D .
 b. Suppose that f is bounded from above by N . Show that if $\lim_{x \rightarrow c} f(x) = L$, then $L \leq N$.
 c. Suppose that f is bounded from below by M . Show that if $\lim_{x \rightarrow c} f(x) = L$, then $L \geq M$.

24. **Max $\{a, b\}$ and min $\{a, b\}$**

- a. Show that the expression

$$\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$$

equals a if $a \geq b$ and equals b if $b \geq a$. In other words, $\max\{a, b\}$ gives the larger of the two numbers a and b .

- b. Find a similar expression for $\min\{a, b\}$, the smaller of a and b .

Generalized Limits Involving $\frac{\sin \theta}{\theta}$

The formula $\lim_{\theta \rightarrow 0} (\sin \theta)/\theta = 1$ can be generalized. If $\lim_{x \rightarrow c} f(x) = 0$ and $f(x)$ is never zero in an open interval containing the point $x = c$, except possibly at c itself, then

$$\lim_{x \rightarrow c} \frac{\sin f(x)}{f(x)} = 1.$$

Here are several examples.

- a. $\lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} = 1$
 b. $\lim_{x \rightarrow 0} \frac{\sin x^2}{x} = \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \lim_{x \rightarrow 0} \frac{x^2}{x} = 1 \cdot 0 = 0$
 c. $\lim_{x \rightarrow -1} \frac{\sin(x^2 - x - 2)}{x + 1}$
 $= \lim_{x \rightarrow -1} \frac{\sin(x^2 - x - 2)}{(x^2 - x - 2)} \cdot \lim_{x \rightarrow -1} \frac{(x^2 - x - 2)}{x + 1}$
 $= 1 \cdot \lim_{x \rightarrow -1} \frac{(x+1)(x-2)}{x+1} = -3$
 d. $\lim_{x \rightarrow 1} \frac{\sin(1 - \sqrt{x})}{x - 1} = \lim_{x \rightarrow 1} \frac{\sin(1 - \sqrt{x})}{1 - \sqrt{x}} \frac{1 - \sqrt{x}}{x - 1}$
 $= \lim_{x \rightarrow 1} \frac{(1 - \sqrt{x})(1 + \sqrt{x})}{(x - 1)(1 + \sqrt{x})}$
 $= \lim_{x \rightarrow 1} \frac{1 - x}{(x - 1)(1 + \sqrt{x})} = -\frac{1}{2}$

Find the limits in Exercises 25–30.

25. $\lim_{x \rightarrow 0} \frac{\sin(1 - \cos x)}{x}$ 26. $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sin \sqrt{x}}$
 27. $\lim_{x \rightarrow 0} \frac{\sin(\sin x)}{x}$ 28. $\lim_{x \rightarrow 0} \frac{\sin(x^2 + x)}{x}$
 29. $\lim_{x \rightarrow 2} \frac{\sin(x^2 - 4)}{x - 2}$ 30. $\lim_{x \rightarrow 9} \frac{\sin(\sqrt{x} - 3)}{x - 9}$

Trigonometric Limits

Find the limits in Exercises 31–38.

31. $\lim_{x \rightarrow 0} \frac{\sin x}{2x^2 - x}$ 32. $\lim_{x \rightarrow 0} \frac{3x - \tan 7x}{2x}$

33. $\lim_{r \rightarrow 0} \frac{\sin r}{\tan 2r}$

34. $\lim_{\theta \rightarrow 0} \frac{\sin(\sin \theta)}{\theta}$

35. $\lim_{\theta \rightarrow (\pi/2)^-} \frac{4 \tan^2 \theta + \tan \theta + 1}{\tan^2 \theta + 5}$

36. $\lim_{\theta \rightarrow 0^+} \frac{1 - 2 \cot^2 \theta}{5 \cot^2 \theta - 7 \cot \theta - 8}$

37. $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}$

38. $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2}$

Show how to extend the functions in Exercises 39 and 40 to be continuous at the origin.

39. $g(x) = \frac{\tan(\tan x)}{\tan x}$

40. $f(x) = \frac{\tan(\tan x)}{\sin(\sin x)}$

Oblique Asymptotes

Find all possible oblique asymptotes in Exercises 41–44.

41. $y = \frac{2x^{3/2} + 2x - 3}{\sqrt{x} + 1}$

42. $y = x + x \sin \frac{1}{x}$

43. $y = \sqrt{x^2 + 1}$

44. $y = \sqrt{x^2 + 2x}$

Showing an Equation Is Solvable

45. Assume that $1 < a < b$ and $\frac{a}{x} + x = \frac{1}{x - b}$. Show that this equation is solvable for x .

More Limits

46. Find constants a and b so that each of the following limits is true.

a. $\lim_{x \rightarrow 0} \frac{\sqrt{a + bx} - 1}{x} = 2$

b. $\lim_{x \rightarrow 1} \frac{\tan(ax - a) + b - 2}{x - 1} = 3$

47. Evaluate $\lim_{x \rightarrow 1} \frac{x^{2/3} - 1}{1 - \sqrt{x}}$.

48. Evaluate $\lim_{x \rightarrow 0} \frac{|3x + 4| - |x| - 4}{x}$.

Limits on Arbitrary Domains

The definition of the limit of a function at $x = c$ extends to functions whose domains near c are more complicated than intervals.

General Definition of Limit

Suppose every open interval containing c contains a point other than c in the domain of f . We say that $\lim_{x \rightarrow c} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x in the domain of f , $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.

For the functions in Exercises 49–52,

- Find the domain.
- Show that at $c = 0$ the domain has the property described above.
- Evaluate $\lim_{x \rightarrow 0} f(x)$.

49. The function f is defined as follows: $f(x) = x$ if $x = 1/n$ where n is a positive integer, and $f(0) = 1$.

50. The function f is defined as follows: $f(x) = 1 - x$ if $x = 1/n$ where n is a positive integer, and $f(0) = 1$.

51. $f(x) = \sqrt{x \sin(1/x)}$

52. $f(x) = \sqrt{\ln(\sin(1/x))}$

53. Let g be a function with domain the rational numbers, defined by $g(x) = \frac{2}{x - \sqrt{2}}$ for rational x .

- Sketch the graph of g as well as you can, keeping in mind that g is defined only at rational points.
- Use the general definition of a limit to prove that $\lim_{x \rightarrow 0} g(x) = -\sqrt{2}$.
- Prove that g is continuous at the point $x = 0$ by showing that the limit in part (b) equals $g(0)$.
- Is g continuous at other points of its domain?

CHAPTER 2 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyLab Math.

Take It to the Limit

Part I

Part II (Zero Raised to the Power Zero: What Does It Mean?)

Part III (One-Sided Limits)

Visualize and interpret the limit concept through graphical and numerical explorations.

Part IV (What a Difference a Power Makes)

See how sensitive limits can be with various powers of x .

Going to Infinity

Part I (Exploring Function Behavior as $x \rightarrow \infty$ or $x \rightarrow -\infty$)

This module provides four examples to explore the behavior of a function as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

Part II (Rates of Growth)

Observe graphs that *appear* to be continuous, yet the function is not continuous. Several issues of continuity are explored to obtain results that you may find surprising.

3

Derivatives



OVERVIEW In Chapter 2 we discussed how to determine the slope of a curve at a point and how to measure the rate at which a function changes. Now that we have studied limits, we can make these notions precise and see that both are interpretations of the *derivative* of a function at a point. We then extend this concept from a single point to the *derivative function*, and we develop rules for finding this derivative function easily, without having to calculate limits directly. These rules are used to find derivatives of most of the common functions reviewed in Chapter 1, as well as combinations of them.

3.1 Tangent Lines and the Derivative at a Point

In this section we define the slope and tangent line to a curve at a point, and the derivative of a function at a point. The derivative gives a way to find both the slope of a graph and the instantaneous rate of change of a function.

Finding a Tangent Line to the Graph of a Function

To find a tangent line to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the procedure introduced in Section 2.1. We calculate the slope of the secant line through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \rightarrow 0$ (Figure 3.1). If the limit exists, we call it the slope of the curve at P and define the tangent line at P to be the line through P having this slope.

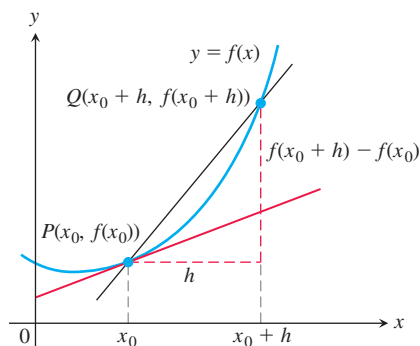


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad (\text{provided the limit exists}).$$

The **tangent line** to the curve at P is the line through P with this slope.

In Section 2.1, Example 3, we applied these definitions to find the slope of the parabola $f(x) = x^2$ at the point $P(2, 4)$ and the tangent line to the parabola at P . Let's look at another example.

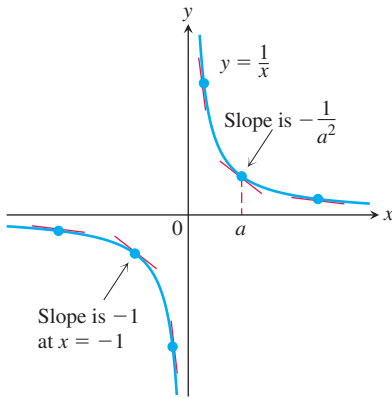


FIGURE 3.2 The tangent lines are steep when x is close to 0, and they become less steep as the point of tangency moves away (Example 1).

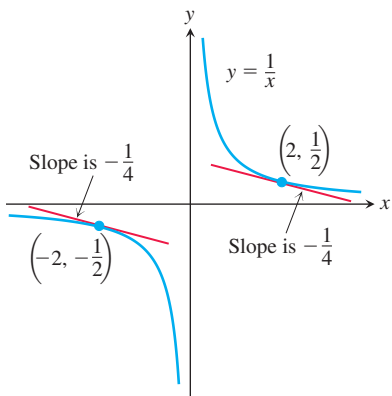


FIGURE 3.3 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 1).

EXAMPLE 1

- (a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?
- (b) Where does the slope equal $-1/4$?
- (c) What happens to the tangent line to the curve at the point $(a, 1/a)$ as a changes?

Solution

- (a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \frac{a - (a+h)}{a(a+h)} \\ &= \lim_{h \rightarrow 0} \frac{-h}{ha(a+h)} = \lim_{h \rightarrow 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.\end{aligned}$$

Notice how we had to keep writing “lim” before each fraction until the stage at which we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 3.2).

- (b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$, provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 3.3).

- (c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \rightarrow 0^+$, the slope approaches $-\infty$ and the tangent line becomes increasingly steep (Figure 3.2). We see this situation again as $a \rightarrow 0^-$. As a moves away from $x = 0$ in either direction, the slope approaches 0 and the tangent line levels off, becoming closer and closer to a horizontal line. ■

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

is called the **difference quotient of f at x_0 with increment h** . If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION The **derivative of a function f at a point x_0** , denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided this limit exists.

The notation $f'(x_0)$ is read “ f prime of x_0 .”

The derivative has more than one meaning, depending on what problem we are considering. The formula for the derivative is the same as the formula for the slope of the curve $y = f(x)$ at a point. If we interpret the difference quotient as the slope of a secant line, then the derivative gives the slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$. If we interpret the difference quotient as an average rate of change (Section 2.1), then the derivative

gives the function's instantaneous rate of change with respect to x at the point $x = x_0$. We study this interpretation in Section 3.4.

EXAMPLE 2 In Examples 1 and 2 in Section 2.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 4.9t^2$ meters during the first t seconds, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. What was the rock's *exact* speed at this time?

Solution We let $f(t) = 4.9t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds, for $h > 0$, was found to be

$$\frac{f(1+h) - f(1)}{h} = \frac{4.9(1+h)^2 - 4.9(1)^2}{h} = \frac{4.9(h^2 + 2h)}{h} = 4.9(h + 2).$$

The rock's speed at the instant $t = 1$ is then

$$f'(1) = \lim_{h \rightarrow 0} 4.9(h + 2) = 4.9(0 + 2) = 9.8 \text{ m/s}.$$

Our original estimate of 9.8 m/s in Section 2.1 was right. ■

Summary

We have been discussing slopes of curves, lines tangent to a curve, the rate of change of a function, and the derivative of a function at a point. All of these ideas are based on the same limit.

All of the following are interpretations for the limit of the difference quotient

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at $x = x_0$

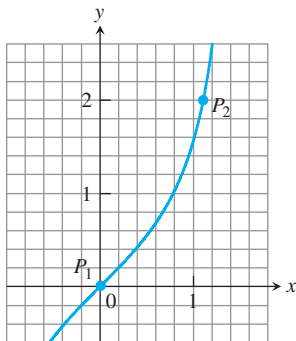
In the next sections, we allow the point x_0 to vary across the domain of the function f .

EXERCISES 3.1

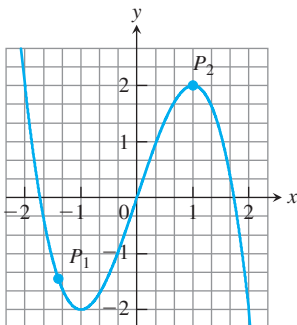
Slopes and Tangent Lines

In Exercises 1–4, use the grid and a straight edge to make a rough estimate of the slope of the curve (in y -units per x -unit) at the points P_1 and P_2 .

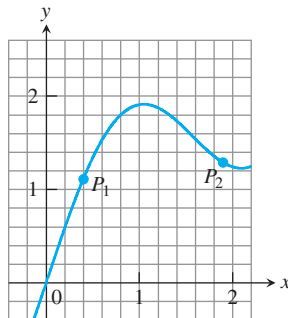
1.



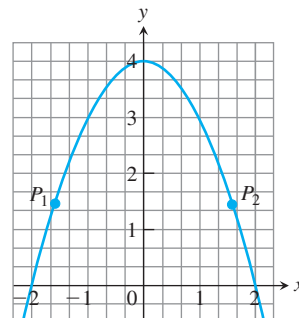
2.



3.



4.



In Exercises 5–10, find an equation for the tangent line to the curve at the given point. Then sketch the curve and tangent line together.

5. $y = 4 - x^2$, $(-1, 3)$ 6. $y = (x - 1)^2 + 1$, $(1, 1)$

7. $y = 2\sqrt{x}$, $(1, 2)$ 8. $y = \frac{1}{x^2}$, $(-1, 1)$

9. $y = x^3$, $(-2, -8)$ 10. $y = \frac{1}{x^3}$, $(-2, -\frac{1}{8})$

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

11. $f(x) = x^2 + 1$, $(2, 5)$ 12. $f(x) = x - 2x^2$, $(1, -1)$

13. $g(x) = \frac{x}{x-2}$, $(3, 3)$ 14. $g(x) = \frac{8}{x^2}$, $(2, 2)$

15. $h(t) = t^3$, $(2, 8)$ 16. $h(t) = t^3 + 3t$, $(1, 4)$

17. $f(x) = \sqrt{x}$, $(4, 2)$ 18. $f(x) = \sqrt{x+1}$, $(8, 3)$

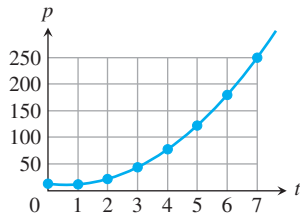
In Exercises 19–22, find the slope of the curve at the point indicated.

19. $y = 5x - 3x^2$, $x = 1$ 20. $y = x^3 - 2x + 7$, $x = -2$

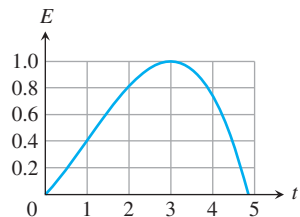
21. $y = \frac{1}{x-1}$, $x = 3$ 22. $y = \frac{x-1}{x+1}$, $x = 0$

Interpreting Derivative Values

23. **Growth of yeast cells** In a controlled laboratory experiment, yeast cells are grown in an automated cell culture system that counts the number P of cells present at hourly intervals. The number after t hours is shown in the accompanying figure.



- Explain what is meant by the derivative $P'(5)$. What are its units?
 - Which is larger, $P'(2)$ or $P'(3)$? Give a reason for your answer.
 - The quadratic curve capturing the trend of the data points (see Appendix A.2) is given by $P(t) = 6.10t^2 - 9.28t + 16.43$. Find the instantaneous rate of growth when $t = 5$ hours.
24. **Effectiveness of a drug** On a scale from 0 to 1, the effectiveness E of a pain-killing drug t hours after entering the bloodstream is displayed in the accompanying figure.



- At what times does the effectiveness appear to be increasing? What is true about the derivative at those times?
- At what time would you estimate that the drug reaches its maximum effectiveness? What is true about the derivative at that time? What is true about the derivative as time increases in the 1 hour before your estimated time?

At what points do the graphs of the functions in Exercises 25 and 26 have horizontal tangent lines?

25. $f(x) = x^2 + 4x - 1$ 26. $g(x) = x^3 - 3x$

27. Find equations of all lines having slope -1 that are tangent to the curve $y = 1/(x-1)$.

28. Find an equation of the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Rates of Change

29. **Object dropped from a tower** An object is dropped from the top of a 100-m-high tower. Its height above ground after t s is $100 - 4.9t^2$ m. How fast is it falling 2 s after it is dropped?

30. **Speed of a rocket** At t seconds after liftoff, the height of a rocket is $3t^2$ m. How fast is the rocket climbing 10 s after liftoff?

31. **Disk's changing area** What is the rate of change of the area of a disk ($A = \pi r^2$) with respect to the radius when the radius is $r = 3$?

32. **Ball's changing volume** What is the rate of change of the volume of a ball ($V = (4/3)\pi r^3$) with respect to the radius when the radius is $r = 2$?

33. Show that the line $y = mx + b$ is its own tangent line at any point $(x_0, mx_0 + b)$.

34. Find the slope of the tangent line to the curve $y = 1/\sqrt{x}$ at the point where $x = 4$.

Testing for Tangent Lines

35. Does the graph of

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent line at the origin? Give reasons for your answer.

36. Does the graph of

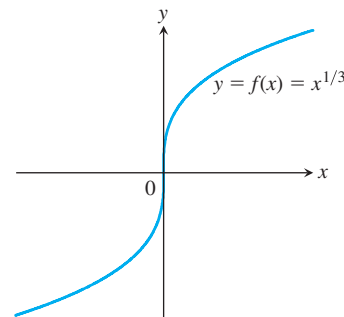
$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent line at the origin? Give reasons for your answer.

Vertical Tangent Lines

We say that a continuous curve $y = f(x)$ has a **vertical tangent line** at the point where $x = x_0$ if the limit of the difference quotient is ∞ or $-\infty$. For example, $y = x^{1/3}$ has a vertical tangent line at $x = 0$ (see accompanying figure):

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0} \frac{h^{1/3} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^{2/3}} = \infty. \end{aligned}$$

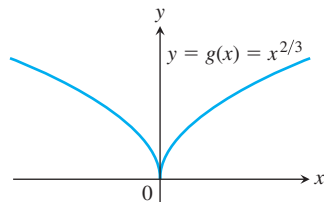


VERTICAL TANGENT LINE AT ORIGIN

However, $y = x^{2/3}$ has *no* vertical tangent line at $x = 0$ (see next figure):

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h} &= \lim_{h \rightarrow 0} \frac{h^{2/3} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^{1/3}}\end{aligned}$$

does not exist, because the limit is ∞ from the right and $-\infty$ from the left.



NO VERTICAL TANGENT LINE AT ORIGIN

37. Does the graph of

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

have a vertical tangent line at the origin? Give reasons for your answer.

38. Does the graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

have a vertical tangent line at the point $(0, 1)$? Give reasons for your answer.

T Graph the curves in Exercises 39–48.

- Where do the graphs appear to have vertical tangent lines?
- Confirm your findings in part (a) with limit calculations. But before you do, read the introduction to Exercises 37 and 38.

39. $y = x^{2/5}$

40. $y = x^{4/5}$

41. $y = x^{1/5}$

42. $y = x^{3/5}$

43. $y = 4x^{2/5} - 2x$

44. $y = x^{5/3} - 5x^{2/3}$

45. $y = x^{2/3} - (x-1)^{1/3}$

46. $y = x^{1/3} + (x-1)^{1/3}$

47. $y = \begin{cases} -\sqrt{|x|}, & x \leq 0 \\ \sqrt{x}, & x > 0 \end{cases}$

48. $y = \sqrt{|4-x|}$

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 49–52:

- Plot $y = f(x)$ over the interval $(x_0 - 1/2) \leq x \leq (x_0 + 3)$.
- Holding x_0 fixed, the difference quotient

$$q(h) = \frac{f(x_0 + h) - f(x_0)}{h}$$

at x_0 becomes a function of the step size h . Enter this function into your CAS workspace.

- Find the limit of q as $h \rightarrow 0$.
- Define the secant lines $y = f(x_0) + q \cdot (x - x_0)$ for $h = 3, 2$, and 1 . Graph them, together with f and the tangent line, over the interval in part (a).

49. $f(x) = x^3 + 2x, \quad x_0 = 0$

50. $f(x) = x + \frac{5}{x}, \quad x_0 = 1$

51. $f(x) = x + \sin(2x), \quad x_0 = \pi/2$

52. $f(x) = \cos x + 4 \sin(2x), \quad x_0 = \pi$

3.2 The Derivative as a Function

HISTORICAL ESSAY

The Derivative

To read this essay, visit the companion Website.

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f .

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

We use the notation $f'(x)$ in the definition, rather than $f'(x_0)$ as before, to emphasize that f' is a function of the independent variable x with respect to which the derivative function $f'(x)$ is being defined. The domain of f' is the set of points in the domain of f for

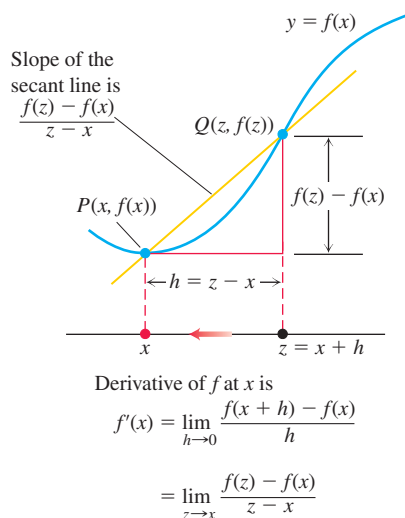


FIGURE 3.4 Two forms for the difference quotient.

which the limit exists, which means that the domain may be the same as or smaller than the domain of f . If f' exists at a particular x , we say that f is **differentiable (has a derivative) at x** . If f' exists at every point in the domain of f , we call f **differentiable**.

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x . Therefore, an equivalent definition of the derivative is as follows (see Figure 3.4). This formula is sometimes more convenient to use when finding a derivative function, and it focuses on the point z that approaches x .

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

as another way to denote the derivative $f'(x)$. Example 1 of Section 3.1 illustrated the differentiation process for the function $y = 1/x$ when $x = a$. For x representing any point in the domain, we get the formula

$$\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}.$$

Here are two more examples in which we allow x to be any point in the domain of f .

EXAMPLE 1 Differentiate $f(x) = \frac{x}{x-1}$.

Solution We use the definition of derivative, which requires us to calculate $f(x+h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x-1} \quad \text{and} \quad f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \text{Definition} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} && \text{Substitute.} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} && \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} && \text{Simplify.} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. && \text{Cancel } h \neq 0 \text{ and evaluate.} \end{aligned}$$

EXAMPLE 2

- (a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.
- (b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Derivative of the Square Root Function

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0$$

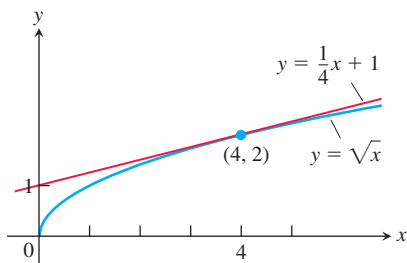


FIGURE 3.5 The curve $y = \sqrt{x}$ and its tangent line at $(4, 2)$. The tangent line's slope is found by evaluating the derivative at $x = 4$ (Example 2).

Solution

(a) We use the alternative formula to calculate f' :

$$\begin{aligned} f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} \\ &= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \quad \frac{1}{a^2 - b^2} = \frac{1}{(a - b)(a + b)} \\ &= \lim_{z \rightarrow x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \quad \text{Cancel and evaluate.} \end{aligned}$$

(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent line is the line through the point $(4, 2)$ with slope $1/4$ (Figure 3.5):

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1.$$

Notation

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y . Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

The symbols d/dx and D indicate the operation of differentiation. We read dy/dx as “the derivative of y with respect to x ,” and df/dx and $(d/dx)f(x)$ as “the derivative of f with respect to x .” The “prime” notations y' and f' originate with Newton. The d/dx notations are similar to those used by Leibniz. The symbol dy/dx should not be regarded as a ratio; it simply denotes a derivative.

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a} = \left. \frac{df}{dx} \right|_{x=a} = \left. \frac{d}{dx}f(x) \right|_{x=a}.$$

For instance, in Example 2,

$$f'(4) = \left. \frac{d}{dx}\sqrt{x} \right|_{x=4} = \left. \frac{1}{2\sqrt{x}} \right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Graphing the Derivative

We can often make an approximate plot of the derivative of $y = f(x)$ by estimating the slopes on the graph of f . That is, we plot the points $(x, f'(x))$ in the xy -plane and connect them with a curve that represents $y = f'(x)$.

EXAMPLE 3 Graph the derivative of the function $y = f(x)$ in Figure 3.6a.

Solution We sketch the tangent lines to the graph of f at frequent intervals and use their slopes to estimate the values of $f'(x)$ at these points. We plot the corresponding $(x, f'(x))$ pairs and connect them with a curve as sketched in Figure 3.6b. ■

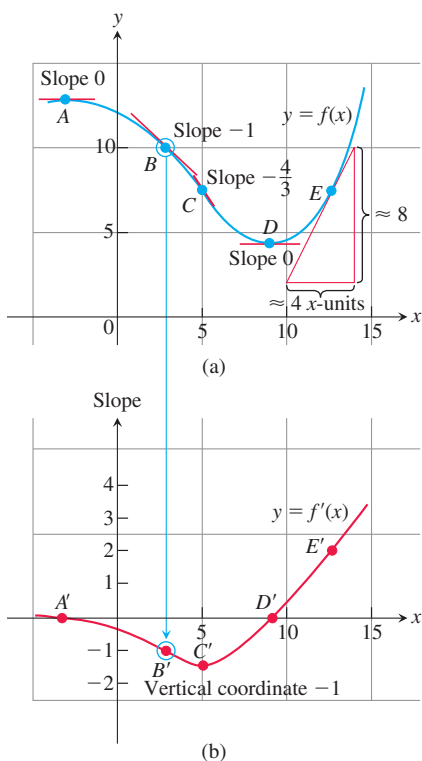


FIGURE 3.6 We made the graph of $y = f'(x)$ in (b) by plotting slopes from the graph of $y = f(x)$ in (a). The vertical coordinate of B' is the slope at B , and so on. The slope at E is approximately $8/4 = 2$. In (b) we see that the rate of change of f is negative for x between A' and D' ; the rate of change is positive for x to the right of D' .

What can we learn from the graph of $y = f'(x)$? At a glance we can see

1. where the rate of change of f is positive, negative, or zero;
2. the rough size of the growth rate at any x ;
3. where the rate of change itself is increasing or decreasing.

Differentiability on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{Right-hand derivative at } a$$

$$\lim_{h \rightarrow 0^-} \frac{f(b+h) - f(b)}{h} \quad \text{Left-hand derivative at } b$$

exist at the endpoints (Figure 3.7).

Right-hand and left-hand derivatives may or may not be defined at any point of a function's domain. Because of Theorem 5, Section 2.4, a function has a derivative at an interior point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

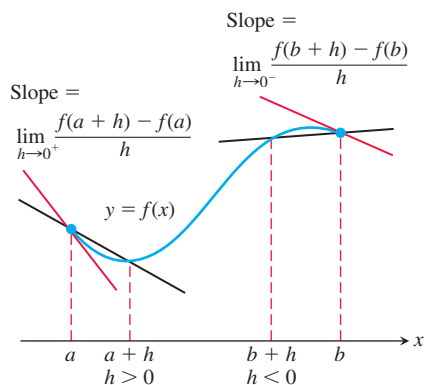


FIGURE 3.7 Derivatives at endpoints of a closed interval are one-sided limits.

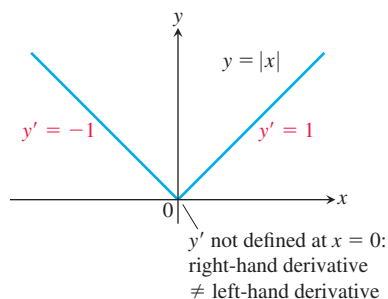


FIGURE 3.8 The function $y = |x|$ is not differentiable at the origin where the graph has a “corner” (Example 4).

EXAMPLE 4 Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and on $(0, \infty)$ but has no derivative at $x = 0$.

Solution The graph of the function $y = mx + b$ is a straight line with slope m . Thus, to the right of the origin, when $x > 0$,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \quad |x| = x \text{ since } x > 0, \quad \frac{d}{dx}(mx+b) = m$$

To the left, when $x < 0$,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1 \quad |x| = -x \text{ since } x < 0$$

(Figure 3.8). The two branches of the graph come together at an angle at the origin, forming a non-smooth corner. There is no derivative at the origin because the one-sided derivatives differ there:

$$\begin{aligned} \text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h \rightarrow 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0^+} \frac{|h|}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{h}{h} \quad |h| = h \text{ when } h > 0 \\ &= \lim_{h \rightarrow 0^+} 1 = 1 \end{aligned}$$

$$\begin{aligned} \text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h \rightarrow 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0^-} \frac{|h|}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{-h}{h} \quad |h| = -h \text{ when } h < 0 \\ &= \lim_{h \rightarrow 0^-} -1 = -1. \end{aligned}$$

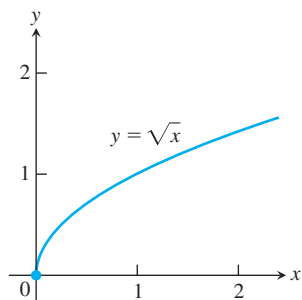


FIGURE 3.9 The square root function is not differentiable at $x = 0$, where the graph of the function has a vertical tangent line.

EXAMPLE 5 In Example 2 we found that for $x > 0$,

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

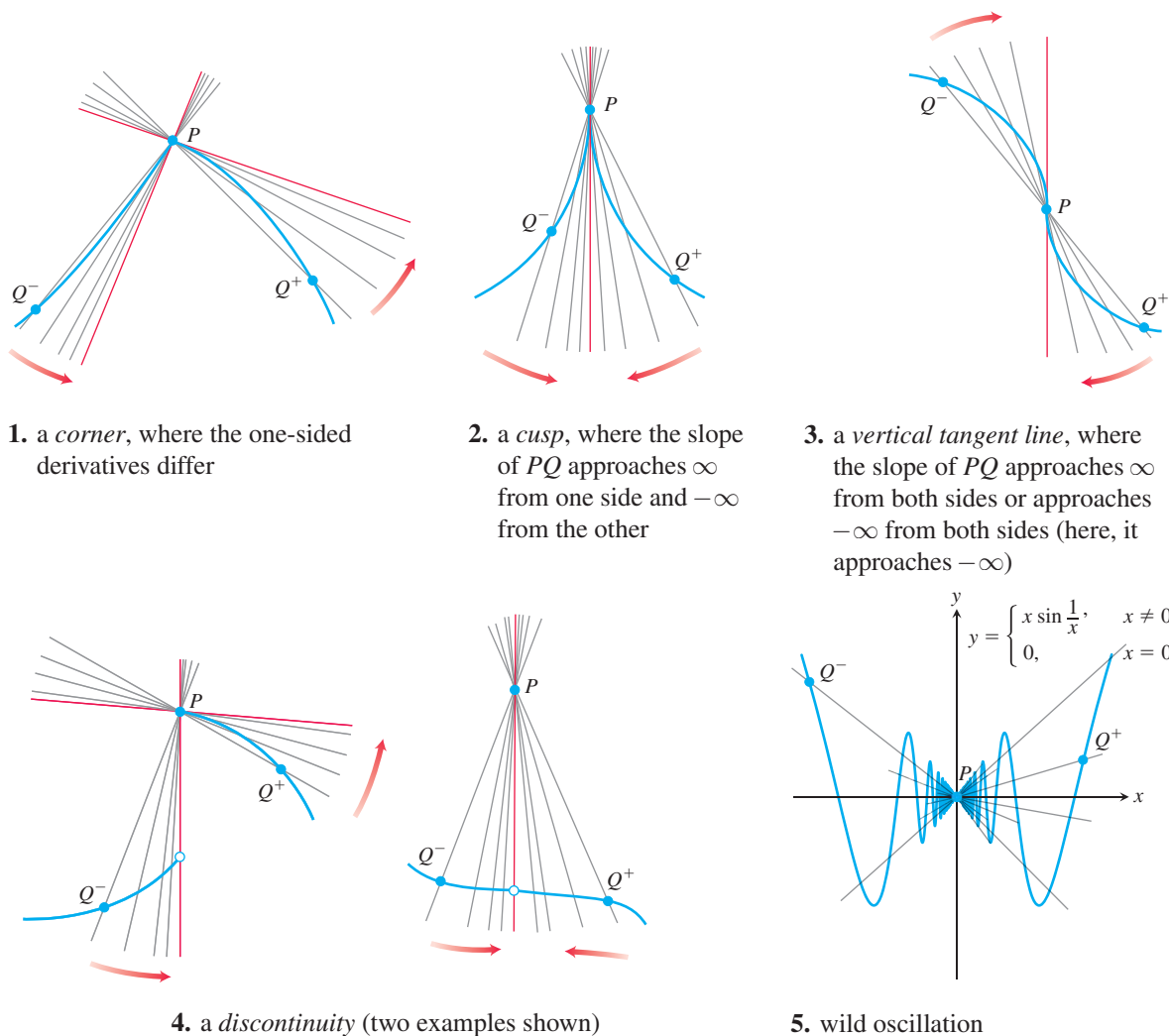
We apply the definition to examine whether the derivative exists at $x = 0$:

$$\lim_{h \rightarrow 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} = \infty.$$

Since the (right-hand) limit is not finite, there is no derivative at $x = 0$. Since the slopes of the secant lines joining the origin to the points $(h, \sqrt{h})$ on a graph of $y = \sqrt{x}$ approach ∞ , the graph has a *vertical tangent line* at the origin. (See Figure 3.9 and Exercises 37 and 38 in Section 3.1.)

When Does a Function Not Have a Derivative at a Point?

A function has a derivative at a point x_0 if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a finite limit as Q approaches P . Thus differentiability is a “smoothness” condition on the graph of f . A function can fail to have a derivative at a point for many reasons, including the existence of points where the graph has



The last example shows a function that is continuous at $x = 0$, but whose graph oscillates wildly up and down as it approaches $x = 0$. The slopes of the secant lines through 0 oscillate between -1 and 1 as x approaches 0 , and do not have a limit at $x = 0$.

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

THEOREM 1—Differentiable Implies Continuous If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof Given that $f'(c)$ exists, we must show that $\lim_{x \rightarrow c} f(x) = f(c)$, or, equivalently, that $\lim_{h \rightarrow 0} f(c + h) = f(c)$. If $h \neq 0$, then

$$\begin{aligned} f(c + h) &= f(c) + (f(c + h) - f(c)) && \text{Add and subtract } f(c). \\ &= f(c) + \frac{f(c + h) - f(c)}{h} \cdot h. && \text{Divide and multiply by } h. \end{aligned}$$

Now take limits as $h \rightarrow 0$. By Theorem 1 of Section 2.2,

$$\begin{aligned} \lim_{h \rightarrow 0} f(c + h) &= \lim_{h \rightarrow 0} f(c) + \lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h} \cdot \lim_{h \rightarrow 0} h \\ &= f(c) + f'(c) \cdot 0 \\ &= f(c) + 0 \\ &= f(c). \end{aligned}$$

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$, then f is continuous from that side at $x = c$.

Theorem 1 says that if a function has a discontinuity at a point (for instance, a jump discontinuity), then it cannot be differentiable there. The greatest integer function $y = \lfloor x \rfloor$ fails to be differentiable at every integer $x = n$ (Example 4, Section 2.6).

Caution The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, as we saw with the absolute value function in Example 4. ●

EXERCISES 3.2

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2$; $f'(-3)$, $f'(0)$, $f'(1)$

2. $F(x) = (x - 1)^2 + 1$; $F'(-1)$, $F'(0)$, $F'(2)$

3. $g(t) = \frac{1}{t^2}$; $g'(-1)$, $g'(2)$, $g'(\sqrt{3})$

4. $k(z) = \frac{1-z}{2z}$; $k'(-1)$, $k'(1)$, $k'(\sqrt{2})$

5. $p(\theta) = \sqrt{3\theta}$; $p'(1)$, $p'(3)$, $p'(2/3)$

6. $r(s) = \sqrt{2s + 1}$; $r'(0)$, $r'(1)$, $r'(1/2)$

In Exercises 7–12, find the indicated derivatives.

7. $\frac{dy}{dx}$ if $y = 2x^3$ 8. $\frac{dr}{ds}$ if $r = s^3 - 2s^2 + 3$

9. $\frac{ds}{dt}$ if $s = \frac{t}{2t + 1}$ 10. $\frac{dv}{dt}$ if $v = t - \frac{1}{t}$

11. $\frac{dp}{dq}$ if $p = q^{3/2}$ 12. $\frac{dz}{dw}$ if $z = \frac{1}{\sqrt{w^2 - 1}}$

Slopes and Tangent Lines

In Exercises 13–16, differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

13. $f(x) = x + \frac{9}{x}$, $x = -3$ 14. $k(x) = \frac{1}{2 + x}$, $x = 2$

15. $s = t^3 - t^2$, $t = -1$ 16. $y = \frac{x + 3}{1 - x}$, $x = -2$

In Exercises 17–18, differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

17. $y = f(x) = \frac{8}{\sqrt{x - 2}}$, $(x, y) = (6, 4)$

18. $w = g(z) = 1 + \sqrt{4 - z}$, $(z, w) = (3, 2)$

In Exercises 19–22, find the values of the derivatives.

19. $\left. \frac{ds}{dt} \right|_{t=-1}$ if $s = 1 - 3t^2$ 20. $\left. \frac{dy}{dx} \right|_{x=\sqrt{3}}$ if $y = 1 - \frac{1}{x}$

21. $\left. \frac{dr}{d\theta} \right|_{\theta=0}$ if $r = \frac{2}{\sqrt{4 - \theta}}$ 22. $\left. \frac{dw}{dz} \right|_{z=4}$ if $w = z + \sqrt{z}$

Using the Alternative Formula for Derivatives

Use the formula

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

to find the derivative of the functions in Exercises 23–26.

23. $f(x) = \frac{1}{x+2}$

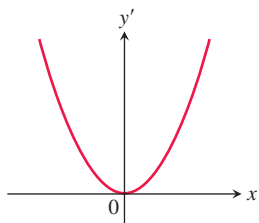
24. $f(x) = x^2 - 3x + 4$

25. $g(x) = \frac{x}{x-1}$

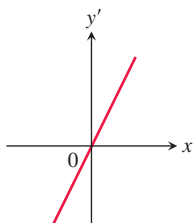
26. $g(x) = 1 + \sqrt{x}$

Graphs

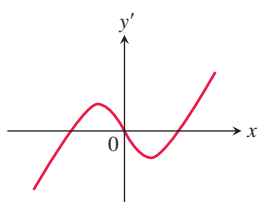
Match the functions graphed in Exercises 27–30 with the derivatives graphed in the accompanying figures (a)–(d).



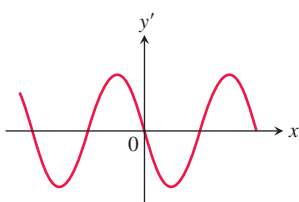
(a)



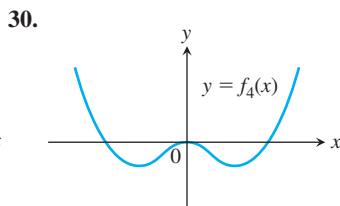
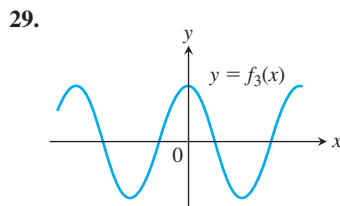
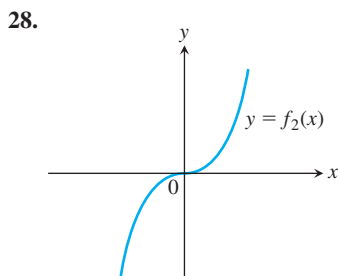
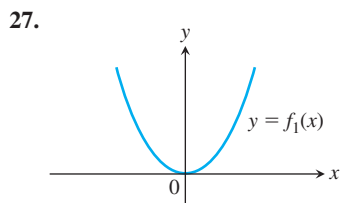
(b)



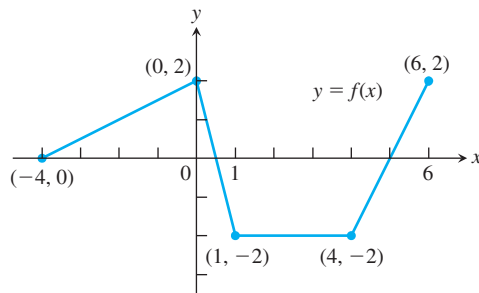
(c)



(d)



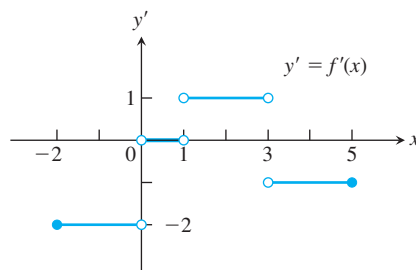
31. Consider the function f graphed here. The domain of f is the interval $[-4, 6]$ and its graph is made of line segments joined end to end.



- a. At which points of the domain interval is f' not defined? Give reasons for your answer.
b. Graph the derivative of f . The graph should show a step function.

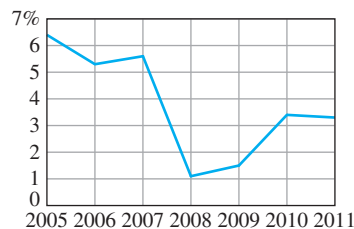
32. Recovering a function from its derivative

- a. Use the following information to graph the function f over the closed interval $[-2, 5]$.
i) The graph of f is made of closed line segments joined end to end.
ii) The graph starts at the point $(-2, 3)$.
iii) The derivative of f is the step function in the figure shown here.



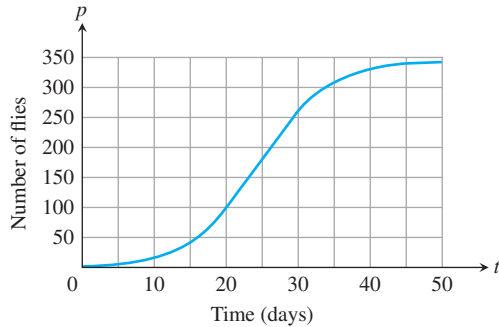
- b. Repeat part (a), assuming that the graph starts at $(-2, 0)$ instead of $(-2, 3)$.

33. **Growth in the economy** The graph in the accompanying figure shows the average annual percentage change $y = f(t)$ in the U.S. gross national product (GNP) for the years 2005–2011. Graph dy/dt (where defined).



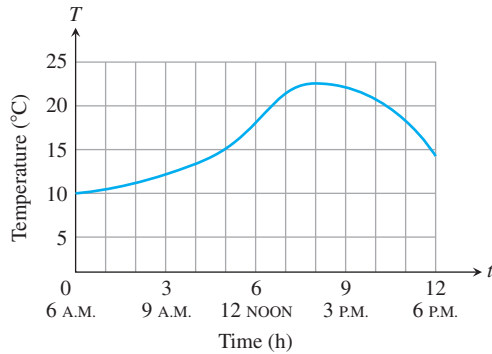
34. **Fruit flies** (Continuation of Example 4, Section 2.1.) Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

- a. Use the graphical technique of Example 3 to graph the derivative of the fruit fly population as a function of time (in days). The graph of the population is reproduced here.



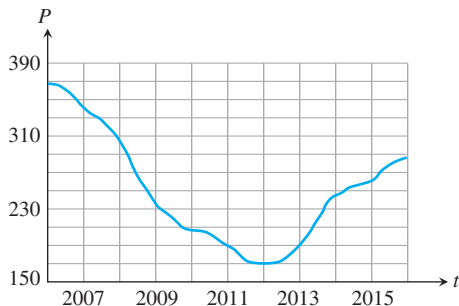
- b. During what days does the population seem to be increasing fastest? Slowest?

35. **Temperature** The given graph shows the temperature T in $^{\circ}\text{C}$ between 6 A.M. and 6 P.M.



- a. Estimate the rate of temperature change at the times
i) 7 A.M. ii) 9 A.M. iii) 2 P.M. iv) 4 P.M.
b. At what time does the temperature increase most rapidly? Decrease most rapidly? What is the rate for each of those times?
c. Use the graphical technique of Example 3 to graph the derivative of temperature T versus time t .

36. Average single-family home prices P (in thousands of dollars) in Sacramento, California, are shown in the accompanying figure from the beginning of 2006 through the end of 2015.



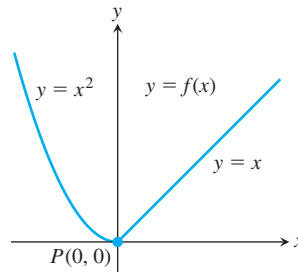
- a. During what years did home prices decrease? increase?
b. Estimate home prices at the end of
i) 2007 ii) 2012 iii) 2015

- c. Estimate the rate of change of home prices at the beginning of
i) 2007 ii) 2010 iii) 2014
d. During what year did home prices drop most rapidly and what is an estimate of this rate?
e. During what year did home prices rise most rapidly and what is an estimate of this rate?
f. Use the graphical technique of Example 3 to graph the derivative of home price P versus time t .

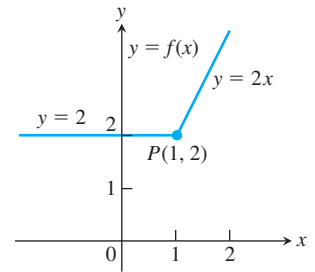
One-Sided Derivatives

Compute the right-hand and left-hand derivatives as limits to show that the functions in Exercises 37–40 are not differentiable at the point P .

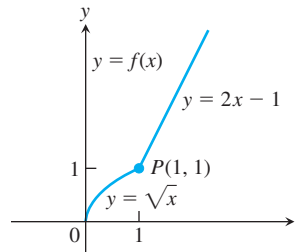
37.



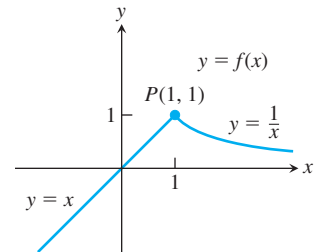
38.



39.



40.



In Exercises 41–44, determine whether the piecewise-defined function is differentiable at $x = 0$.

$$41. f(x) = \begin{cases} 2x - 1, & x \geq 0 \\ x^2 + 2x + 7, & x < 0 \end{cases}$$

$$42. g(x) = \begin{cases} x^{2/3}, & x \geq 0 \\ x^{1/3}, & x < 0 \end{cases}$$

$$43. f(x) = \begin{cases} 2x + \tan x, & x \geq 0 \\ x^2, & x < 0 \end{cases}$$

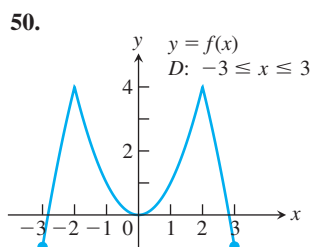
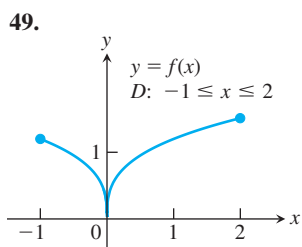
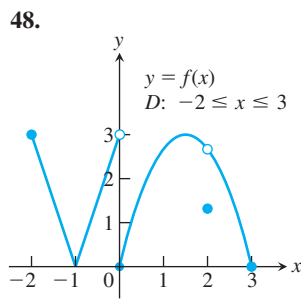
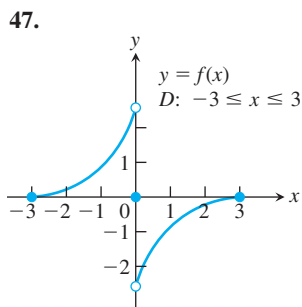
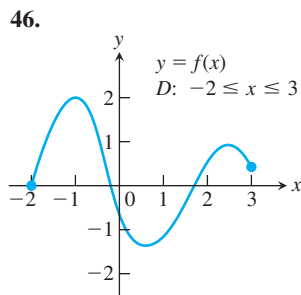
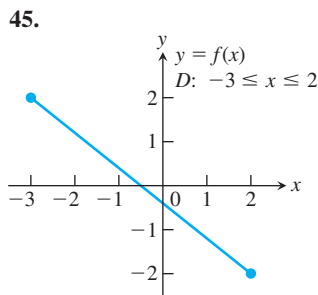
$$44. g(x) = \begin{cases} 2x - x^3 - 1, & x \geq 0 \\ x - \frac{1}{x+1}, & x < 0 \end{cases}$$

Differentiability and Continuity on an Interval

Each figure in Exercises 45–50 shows the graph of a function over a closed interval D . At what domain points does the function appear to be

- a. differentiable?
b. continuous but not differentiable?
c. neither continuous nor differentiable?

Give reasons for your answers.



Theory and Examples

In Exercises 51–54,

- Find the derivative $f'(x)$ of the given function $y = f(x)$.
 - Graph $y = f(x)$ and $y = f'(x)$ side by side using separate sets of coordinate axes, and answer the following questions.
 - For what values of x , if any, is f' positive? Zero? Negative?
 - Over what intervals of x -values, if any, does the function $y = f(x)$ increase as x increases? Decrease as x increases? How is this related to what you found in part (c)? (We will say more about this relationship in Section 4.3.)
- $y = -x^2$
 - $y = -1/x$
 - $y = x^3/3$
 - $y = x^4/4$
- Tangent line to a parabola** Does the parabola $y = 2x^2 - 13x + 5$ have a tangent line whose slope is -1 ? If so, find an equation for the line and the point of tangency. If not, why not?
 - Tangent line to $y = \sqrt{x}$** Does any tangent line to the curve $y = \sqrt{x}$ cross the x -axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?
 - Derivative of $-f$** Does knowing that a function $f(x)$ is differentiable at $x = x_0$ tell you anything about the differentiability of the function $-f$ at $x = x_0$? Give reasons for your answer.
 - Derivative of multiples** Does knowing that a function $g(t)$ is differentiable at $t = 7$ tell you anything about the differentiability of the function $3g$ at $t = 7$? Give reasons for your answer.

59. Limit of a quotient Suppose that functions $g(t)$ and $h(t)$ are defined for all values of t and $g(0) = h(0) = 0$. Can $\lim_{t \rightarrow 0} g(t)/h(t)$ exist? If it does exist, must it equal zero? Give reasons for your answers.

- Let $f(x)$ be a function satisfying $|f(x)| \leq x^2$ for $-1 \leq x \leq 1$. Show that f is differentiable at $x = 0$ and find $f'(0)$.
- Show that

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ and find $f'(0)$.

- T 61.** Graph $y = 1/(2\sqrt{x})$ in a window that has $0 \leq x \leq 2$. Then, on the same screen, graph

$$y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

for $h = 1, 0.5, 0.1$. Then try $h = -1, -0.5, -0.1$. Explain what is going on.

- T 62.** Graph $y = 3x^2$ in a window that has $-2 \leq x \leq 2, 0 \leq y \leq 3$. Then, on the same screen, graph

$$y = \frac{(x+h)^3 - x^3}{h}$$

for $h = 2, 1, 0.2$. Then try $h = -2, -1, -0.2$. Explain what is going on.

- 63. Derivative of $y = |x|$** Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

- T 64. Weierstrass's nowhere differentiable continuous function** The sum of the first eight terms of the Weierstrass function $f(x) = \sum_{n=0}^{\infty} (2/3)^n \cos(9^n \pi x)$ is

$$g(x) = \cos(\pi x) + (2/3)^1 \cos(9\pi x) + (2/3)^2 \cos(9^2 \pi x) + (2/3)^3 \cos(9^3 \pi x) + \cdots + (2/3)^7 \cos(9^7 \pi x).$$

Graph this sum. Zoom in several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 65–70.

- Plot $y = f(x)$ to see that function's global behavior.
- Define the difference quotient q at a general point x , with general step size h .
- Take the limit as $h \rightarrow 0$. What formula does this give?
- Substitute the value $x = x_0$ and plot the function $y = f(x)$ together with its tangent line at that point.
- Substitute various values for x larger and smaller than x_0 into the formula obtained in part (c). Do the numbers make sense with your picture?
- Graph the formula obtained in part (c). What does it mean when its values are negative? Zero? Positive? Does this make sense with your plot from part (a)? Give reasons for your answer.

65. $f(x) = x^3 + x^2 - x, \quad x_0 = 1$

66. $f(x) = x^{1/3} + x^{2/3}, \quad x_0 = 1$

67. $f(x) = \frac{4x}{x^2 + 1}, \quad x_0 = 2$

68. $f(x) = \frac{x-1}{3x^2+1}, \quad x_0 = -1$

69. $f(x) = \sin 2x, \quad x_0 = \pi/2$

70. $f(x) = x^2 \cos x, \quad x_0 = \pi/4$

3.3 Differentiation Rules

This section introduces several rules that allow us to differentiate constant functions, power functions, polynomials, exponential functions, rational functions, and certain combinations of them, simply and directly, without having to take limits each time.

Powers, Multiples, Sums, and Differences

A basic rule of differentiation is that the derivative of every constant function is zero.

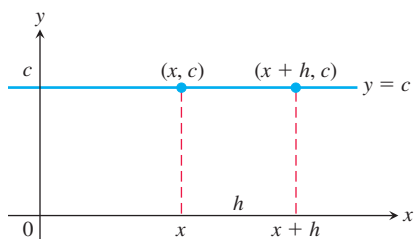


FIGURE 3.10 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Proof We apply the definition of the derivative to $f(x) = c$, the function whose outputs have the constant value c (Figure 3.10). At every value of x , we find that

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0. \quad \blacksquare$$

We now consider powers of x . From Section 3.1, we know that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad \text{or} \quad \frac{d}{dx}(x^{-1}) = -x^{-2}.$$

From Example 2 of the last section we also know that

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}, \quad \text{or} \quad \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}.$$

These two examples illustrate a general rule for differentiating a power x^n . We first prove the rule when n is a positive integer.

Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof of the Positive Integer Power Rule The formula

$$z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$$

can be verified by multiplying out the right-hand side. Then, from the alternative formula for the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} = \lim_{z \rightarrow x} \frac{z^n - x^n}{z - x} \\ &= \lim_{z \rightarrow x} (z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}) \quad n \text{ terms} \\ &= nx^{n-1}. \quad \blacksquare \end{aligned}$$

HISTORICAL BIOGRAPHY

Richard Courant
(1888–1972)

Courant obtained a PhD from Göttingen in 1910. He founded Göttingen's Mathematics Institute and was its director from 1920 until 1933. His research work focused on mathematical physics.

To know more, visit the companion Website.

The Power Rule is actually valid for all real numbers n , not just for positive integers. We have seen examples for a negative integer and fractional power, but n could be an irrational number as well. Here we state the general version of the rule, but postpone its proof until Section 3.8.

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

EXAMPLE 1 Differentiate the following powers of x .

(a) x^3 (b) $x^{2/3}$ (c) $x^{\sqrt{2}}$ (d) $\frac{1}{x^4}$ (e) $x^{-4/3}$ (f) $\sqrt{x^{2+\pi}}$

Solution

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(x^3) &= 3x^{3-1} = 3x^2 \\ \text{(b)} \quad \frac{d}{dx}(x^{2/3}) &= \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3} \\ \text{(c)} \quad \frac{d}{dx}(x^{\sqrt{2}}) &= \sqrt{2}x^{\sqrt{2}-1} \\ \text{(d)} \quad \frac{d}{dx}\left(\frac{1}{x^4}\right) &= \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5} \\ \text{(e)} \quad \frac{d}{dx}(x^{-4/3}) &= -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3} \\ \text{(f)} \quad \frac{d}{dx}(\sqrt{x^{2+\pi}}) &= \frac{d}{dx}(x^{1+(\pi/2)}) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^\pi} \end{aligned}$$

The next rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

Derivative Constant Multiple Rule

If u is a differentiable function of x , and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Proof

$$\begin{aligned} \frac{d}{dx}cu &= \lim_{h \rightarrow 0} \frac{cu(x+h) - cu(x)}{h} && \text{Derivative definition} \\ &&& \text{with } f(x) = cu(x) \\ &= c \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} && \text{Constant Multiple Rule for Limits} \\ &= c \frac{du}{dx} && u \text{ is differentiable.} \end{aligned}$$

Applying the Power Rule

Subtract 1 from the exponent and multiply the result by the original exponent.

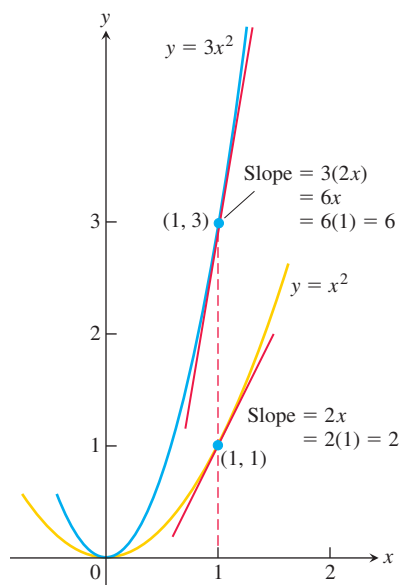


FIGURE 3.11 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinate triples the slope (Example 2).

EXAMPLE 2

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.11).

(b) Negative of a function

The derivative of the negative of a differentiable function u is the negative of the function's derivative. The Constant Multiple Rule with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

Derivative Sum Rule

If u and v are differentiable functions of x , then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Denoting Functions by u and v

The functions we are working with when we need a differentiation formula are likely to be denoted by letters like f and g . We do not want to use these same letters when stating general differentiation rules, so instead we use letters like u and v that are not likely to be already in use.

Proof We apply the definition of the derivative to $f(x) = u(x) + v(x)$:

$$\begin{aligned} \frac{d}{dx}[u(x) + v(x)] &= \lim_{h \rightarrow 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}. \end{aligned}$$

Combining the Sum Rule with the Constant Multiple Rule gives the **Difference Rule**, which says that the derivative of a *difference* of differentiable functions is the difference of their derivatives:

$$\frac{d}{dx}(u - v) = \frac{d}{dx}[u + (-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}.$$

The Sum Rule also extends to finite sums of more than two functions. If $u_1, u_2, \dots, u_n$ are differentiable at x , then so is $u_1 + u_2 + \dots + u_n$, and

$$\frac{d}{dx}(u_1 + u_2 + \dots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \dots + \frac{du_n}{dx}.$$

A proof by mathematical induction for any finite number of terms is given in Appendix A.3.

EXAMPLE 3 Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

Solution $\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$ Sum and Difference Rules

$$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

We can differentiate any polynomial term by term, the way we differentiated the polynomial in Example 3. All polynomials are differentiable at all values of x .

EXAMPLE 4 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangent lines? If so, where?

Solution The horizontal tangent lines, if any, occur where the slope dy/dx is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for x :

$$\begin{aligned} 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ x &= 0, 1, -1. \end{aligned}$$

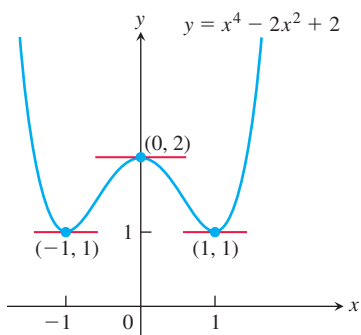


FIGURE 3.12 The curve in Example 4 and its horizontal tangent lines.

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangent lines at $x = 0, 1$, and -1 . The corresponding points on the curve are $(0, 2)$, $(1, 1)$, and $(-1, 1)$. See Figure 3.12.

Derivatives of Exponential Functions

We briefly reviewed exponential functions in Section 1.4. When we apply the definition of the derivative to $f(x) = a^x$, we get

$$\begin{aligned} \frac{d}{dx}(a^x) &= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} && \text{Derivative definition} \\ &= \lim_{h \rightarrow 0} \frac{a^x \cdot a^h - a^x}{h} && a^{x+h} = a^x \cdot a^h \\ &= \lim_{h \rightarrow 0} a^x \cdot \frac{a^h - 1}{h} && \text{Factoring out } a^x \\ &= a^x \cdot \lim_{h \rightarrow 0} \frac{a^h - 1}{h} && a^x \text{ is constant as } h \rightarrow 0. \\ &= \left(\lim_{h \rightarrow 0} \frac{a^h - 1}{h} \right) \cdot a^x. && (1) \end{aligned}$$

a fixed number L

Thus we see that the derivative of a^x is a constant multiple L of a^x . The constant L is a limit we have not encountered before. Note, however, that it equals the derivative of $f(x) = a^x$ at $x = 0$:

$$f'(0) = \lim_{h \rightarrow 0} \frac{a^h - a^0}{h} = \lim_{h \rightarrow 0} \frac{a^h - 1}{h} = L.$$

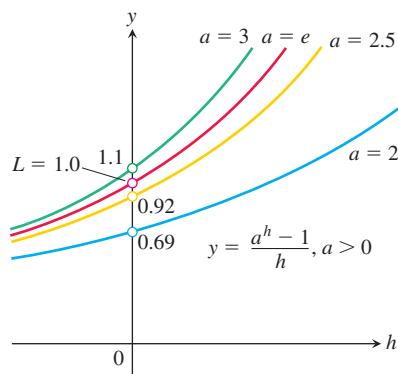


FIGURE 3.13 The position of the curve $y = (a^h - 1)/h$, $a > 0$, varies continuously with a . The limit L of y as $h \rightarrow 0$ changes with different values of a . The number for which $L = 1$ as $h \rightarrow 0$ is the number e between $a = 2$ and $a = 3$.

The limit L is therefore the slope of the graph of $f(x) = a^x$ where it crosses the y -axis. In Chapter 7, where we carefully develop the logarithmic and exponential functions, we prove that the limit L exists and has the value $\ln a$. For now we investigate values of L by graphing the function $y = (a^h - 1)/h$ and studying its behavior as h approaches 0.

Figure 3.13 shows the graphs of $y = (a^h - 1)/h$ for four different values of a . The limit L is approximately 0.69 if $a = 2$, about 0.92 if $a = 2.5$, and about 1.1 if $a = 3$. It appears that the value of L is 1 at some number a chosen between 2.5 and 3. That number is given by $a = e \approx 2.718281828$. With this choice of base we obtain the natural exponential function $f(x) = e^x$ as in Section 1.4, and see that it satisfies the property

$$f'(0) = \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1 \quad (2)$$

because it is the exponential function whose graph has slope 1 when it crosses the y -axis. That the limit is 1 implies an important relationship between the natural exponential function e^x and its derivative:

$$\begin{aligned} \frac{d}{dx}(e^x) &= \lim_{h \rightarrow 0} \left(\frac{e^h - 1}{h} \right) \cdot e^x && \text{Eq. (1) with } a = e \\ &= 1 \cdot e^x = e^x. && \text{Eq. (2)} \end{aligned}$$

Therefore the natural exponential function is its own derivative.

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

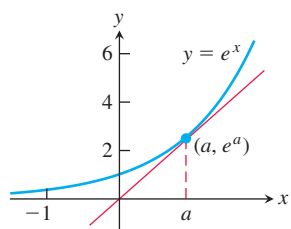


FIGURE 3.14 The line through the origin is tangent to the graph of $y = e^x$ when $a = 1$ (Example 5).

EXAMPLE 5 Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

Solution Since the line passes through the origin, its equation is of the form $y = mx$, where m is the slope. If it is tangent to the graph at the point (a, e^a) , the slope is $m = (e^a - 0)/(a - 0)$. The slope of the natural exponential at $x = a$ is e^a . Because these slopes are the same, we then have that $e^a = e^a/a$. It follows that $a = 1$ and $m = e$, so the equation of the tangent line is $y = ex$. See Figure 3.14. ■

We might ask if there are functions *other* than the natural exponential function that are their own derivatives. The answer is that the only functions that satisfy the property that $f'(x) = f(x)$ are functions that are constant multiples of the natural exponential function, $f(x) = c \cdot e^x$, c any constant. We prove this fact in Section 7.2. Note from the Constant Multiple Rule that indeed

$$\frac{d}{dx}(c \cdot e^x) = c \cdot \frac{d}{dx}(e^x) = c \cdot e^x.$$

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \quad \text{while} \quad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

Derivative Product Rule

If u and v are differentiable at x , then so is their product uv , and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + \frac{du}{dx} v.$$

The derivative of the product uv is u times the derivative of v plus the derivative of u times v . In prime notation, $(uv)' = uv' + u'v$. In function notation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x), \quad \text{or} \quad (fg)' = fg' + f'g. \quad (3)$$

EXAMPLE 6 Find the derivative of (a) $y = \frac{1}{x}(x^2 + e^x)$, (b) $y = e^{2x}$.

Solution

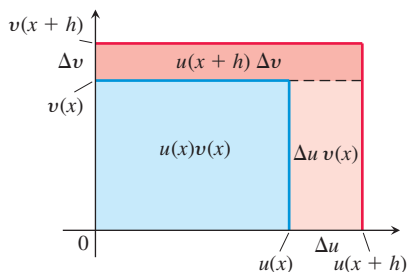
(a) We apply the Product Rule with $u = 1/x$ and $v = x^2 + e^x$:

$$\begin{aligned} \frac{d}{dx}\left[\frac{1}{x}(x^2 + e^x)\right] &= \frac{1}{x}(2x + e^x) + \left(-\frac{1}{x^2}\right)(x^2 + e^x) && \frac{d}{dx}(uv) = u \frac{dv}{dx} + \frac{du}{dx} v \text{ and} \\ &= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2} && \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2} \\ &= 1 + (x - 1) \frac{e^x}{x^2}. \end{aligned}$$

$$(b) \frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot \frac{d}{dx}(e^x) + \frac{d}{dx}(e^x) \cdot e^x = 2e^x \cdot e^x = 2e^{2x} \quad \blacksquare$$

Picturing the Product Rule

Suppose $u(x)$ and $v(x)$ are positive and increase when x increases, and $h > 0$.



Then the change in the product uv is the difference in areas of the larger and smaller “boxes,” which is the sum of the areas of the upper and right-hand reddish-shaded rectangles. That is,

$$\begin{aligned} \Delta(uv) &= u(x+h)v(x+h) - u(x)v(x) \\ &= u(x+h)\Delta v + \Delta u v(x). \end{aligned}$$

Division by h gives

$$\frac{\Delta(uv)}{h} = u(x+h) \frac{\Delta v}{h} + \frac{\Delta u}{h} v(x).$$

The limit as $h \rightarrow 0^+$ yields the Product Rule.

Proof of the Derivative Product Rule

$$\frac{d}{dx}(uv) = \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of u and v , we subtract and add $u(x+h)v(x)$ in the numerator:

$$\begin{aligned} \frac{d}{dx}(uv) &= \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\ &= \lim_{h \rightarrow 0} \left[u(x+h) \frac{v(x+h) - v(x)}{h} + \frac{u(x+h) - u(x)}{h} v(x) \right] \\ &= \lim_{h \rightarrow 0} u(x+h) \cdot \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} + \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} \cdot v(x). \end{aligned}$$

As h approaches zero, $u(x+h)$ approaches $u(x)$ because u , being differentiable at x , is continuous at x . The two fractions approach the values of dv/dx at x and du/dx at x . Therefore,

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + \frac{du}{dx} v. \quad \blacksquare$$

The derivative of the quotient of two functions is given by the Quotient Rule.

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x , and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

In function notation,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

EXAMPLE 7 Find the derivative of (a) $y = \frac{t^2 - 1}{t^3 + 1}$, (b) $y = e^{-x}$.

Solution

(a) We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\begin{aligned} \frac{dy}{dt} &= \frac{(t^3 + 1) \cdot 2t - (t^2 - 1) \cdot 3t^2}{(t^3 + 1)^2} & \frac{d}{dt}\left(\frac{u}{v}\right) &= \frac{v(du/dt) - u(dv/dt)}{v^2} \\ &= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2} \\ &= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}. \end{aligned}$$

$$(b) \frac{d}{dx}(e^{-x}) = \frac{d}{dx}\left(\frac{1}{e^x}\right) = \frac{e^x \cdot 0 - 1 \cdot e^x}{(e^x)^2} = \frac{-1}{e^x} = -e^{-x}$$

Proof of the Derivative Quotient Rule

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h \rightarrow 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{h v(x+h)v(x)} \end{aligned}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v , we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h \rightarrow 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{h v(x+h)v(x)} \\ &= \lim_{h \rightarrow 0} \frac{v(x) \frac{u(x+h) - u(x)}{h} - u(x) \frac{v(x+h) - v(x)}{h}}{v(x+h)v(x)}. \end{aligned}$$

Taking the limits in the numerator and denominator now gives the Quotient Rule. Exercise 76 outlines another proof. ■

The choice of which rules to use in solving a differentiation problem can make a difference in how much work you have to do. Here is an example.

HISTORICAL BIOGRAPHY

Maria Gaetana Agnesi
(1718–1799)

Agnesi was a well-published scientist by age 20 and an honorary faculty member of the University of Bologna by age 30. Today, Agnesi is remembered chiefly for a bell-shaped curve called the “Witch of Agnesi.”

To know more, visit the companion Website.

EXAMPLE 8 Find the derivative of

$$y = \frac{(x-1)(x^2-2x)}{x^4}.$$

Solution Using the Quotient Rule here will result in a complicated expression with many terms. Instead, use some algebra to simplify the expression. First expand the numerator and divide by x^4 :

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3-3x^2+2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum, Constant Multiple, and Power Rules:

$$\begin{aligned}\frac{dy}{dx} &= -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} \\ &= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.\end{aligned}$$

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f'' . So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

The symbol D^2 means that the operation of differentiation is performed twice.

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30x^4.$$

Thus $D^2(x^6) = 30x^4$.

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of y with respect to x . The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y$$

denoting the **n th derivative** of y with respect to x for any positive integer n .

We can interpret the second derivative as the rate of change of the slope of the tangent line to the graph of $y = f(x)$ at each point. You will see in the next chapter that the second derivative reveals whether the graph bends upward or downward from the tangent line as we move off the point of tangency. In the next section, we interpret both the second and third derivatives in terms of motion along a straight line.

EXAMPLE 9 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

$$\begin{aligned}\text{First derivative:} & \quad y' = 3x^2 - 6x \\ \text{Second derivative:} & \quad y'' = 6x - 6 \\ \text{Third derivative:} & \quad y''' = 6 \\ \text{Fourth derivative:} & \quad y^{(4)} = 0.\end{aligned}$$

All polynomial functions have derivatives of all orders. In this example, the fifth and later derivatives are all zero.

How to Read the Symbols for Derivatives

y' “y prime”

y'' “y double prime”

$\frac{d^2y}{dx^2}$ “d squared y by dx squared”

y''' “y triple prime”

$y^{(n)}$ “y super n ”

$\frac{d^n y}{dx^n}$ “d to the n of y by dx to the n ”

D^n “d to the n ”

EXERCISES 3.3

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$
2. $y = x^2 + x + 8$
3. $s = 5t^3 - 3t^5$
4. $w = 3z^7 - 7z^3 + 21z^2$
5. $y = \frac{4x^3}{3} - x + 2e^x$
6. $y = \frac{x^3}{3} + \frac{x^2}{2} + e^{-x}$
7. $w = 3z^{-2} - \frac{1}{z}$
8. $s = -2t^{-1} + \frac{4}{t^2}$
9. $y = 6x^2 - 10x - 5x^{-2}$
10. $y = 4 - 2x - x^{-3}$
11. $r = \frac{1}{3s^2} - \frac{5}{2s}$
12. $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

In Exercises 13–16, find y' (a) by applying the Product Rule and (b) by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y = (3 - x^2)(x^3 - x + 1)$
14. $y = (2x + 3)(5x^2 - 4x)$
15. $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$
16. $y = (1 + x^2)(x^{3/4} - x^{-3})$

Find the derivatives of the functions in Exercises 17–40.

17. $y = \frac{2x + 5}{3x - 2}$
18. $z = \frac{4 - 3x}{3x^2 + x}$
19. $g(x) = \frac{x^2 - 4}{x + 0.5}$
20. $f(t) = \frac{t^2 - 1}{t^2 + t - 2}$
21. $v = (1 - t)(1 + t^2)^{-1}$
22. $w = (2x - 7)^{-1}(x + 5)$
23. $f(s) = \frac{\sqrt{s} - 1}{\sqrt{s} + 1}$
24. $u = \frac{5x + 1}{2\sqrt{x}}$
25. $v = \frac{1 + x - 4\sqrt{x}}{x}$
26. $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$
27. $y = \frac{1}{(x^2 - 1)(x^2 + x + 1)}$
28. $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$
29. $y = 2e^{-x} + e^{3x}$
30. $y = \frac{x^2 + 3e^x}{2e^x - x}$
31. $y = x^3e^x$
32. $w = re^{-r}$
33. $y = x^{9/4} + e^{-2x}$
34. $y = x^{-3/5} + \pi^{3/2}$
35. $s = 2t^{3/2} + 3e^2$
36. $w = \frac{1}{z^{1.4}} + \frac{\pi}{\sqrt{z}}$
37. $y = \sqrt[3]{x^2} - x^e$
38. $y = \sqrt[3]{x^{9.6}} + 2e^{1.3}$
39. $r = \frac{e^s}{s}$
40. $r = e^{\theta}\left(\frac{1}{\theta^2} + \theta^{-\pi/2}\right)$

Find the derivatives of all orders of the functions in Exercises 41–44.

41. $y = \frac{x^4}{2} - \frac{3}{2}x^2 - x$
42. $y = \frac{x^5}{120}$
43. $y = (x - 1)(x + 2)(x + 3)$
44. $y = (4x^2 + 3)(2 - x)x$

Find the first and second derivatives of the functions in Exercises 45–52.

45. $y = \frac{x^3 + 7}{x}$
46. $s = \frac{t^2 + 5t - 1}{t^2}$
47. $r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$
48. $u = \frac{(x^2 + x)(x^2 - x + 1)}{x^4}$

$$49. w = \left(\frac{1 + 3z}{3z}\right)(3 - z) \quad 50. p = \frac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$$

$$51. w = 3z^2e^{2z} \quad 52. w = e^z(z - 1)(z^2 + 1)$$

53. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

$$\text{a. } \frac{d}{dx}(uv) \quad \text{b. } \frac{d}{dx}\left(\frac{u}{v}\right) \quad \text{c. } \frac{d}{dx}\left(\frac{v}{u}\right) \quad \text{d. } \frac{d}{dx}(7v - 2u)$$

54. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

$$\text{a. } \frac{d}{dx}(uv) \quad \text{b. } \frac{d}{dx}\left(\frac{u}{v}\right) \quad \text{c. } \frac{d}{dx}\left(\frac{v}{u}\right) \quad \text{d. } \frac{d}{dx}(7v - 2u)$$

Slopes and Tangent Lines

55. a. **Normal line to a curve** Find an equation for the line perpendicular to the tangent line to the curve $y = x^3 - 4x + 1$ at the point $(2, 1)$.

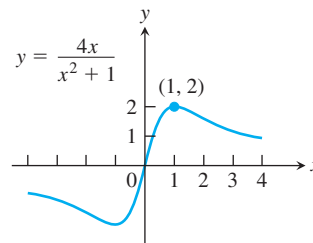
b. **Smallest slope** What is the smallest slope on the curve? At what point on the curve does the curve have this slope?

c. **Tangent lines having specified slope** Find equations for the tangent lines to the curve at the points where the slope of the curve is 8.

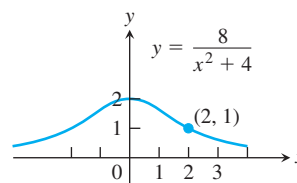
56. a. **Horizontal tangent lines** Find equations for the horizontal tangent lines to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangent lines at the points of tangency.

b. **Smallest slope** What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent line at this point.

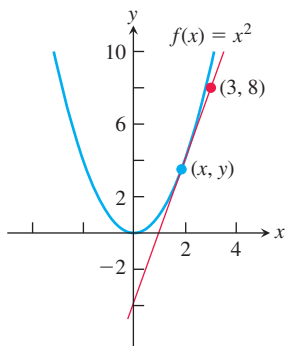
57. Find the tangent lines to *Newton's serpentine* (graphed here) at the origin and the point $(1, 2)$.



58. Find the tangent line to the *Witch of Agnesi* (graphed here) at the point $(2, 1)$.



- 59. Quadratic tangent to identity function** The curve $y = ax^2 + bx + c$ passes through the point $(1, 2)$ and is tangent to the line $y = x$ at the origin. Find a , b , and c .
- 60. Quadratics having a common tangent line** The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point $(1, 0)$. Find a , b , and c .
- 61.** Find all points (x, y) on the graph of $f(x) = 3x^2 - 4x$ with tangent lines parallel to the line $y = 8x + 5$.
- 62.** Find all points (x, y) on the graph of $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 1$ with tangent lines parallel to the line $8x - 2y = 1$.
- 63.** Find all points (x, y) on the graph of $y = x/(x - 2)$ with tangent lines perpendicular to the line $y = 2x + 3$.
- 64.** Find all points (x, y) on the graph of $f(x) = x^2$ with tangent lines passing through the point $(3, 8)$.



- 65.** Assume that functions f and g are differentiable with $f(1) = 2$, $f'(1) = -3$, $g(1) = 4$, and $g'(1) = -2$. Find the equation of the line tangent to the graph of $F(x) = f(x)g(x)$ at $x = 1$.
- 66.** Assume that functions f and g are differentiable with $f(2) = 3$, $f'(2) = -1$, $g(2) = -4$, and $g'(2) = 1$. Find an equation of the line perpendicular to the line tangent to the graph of $F(x) = \frac{f(x) + 3}{x - g(x)}$ at $x = 2$.
- 67. a.** Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$.
- T b.** Graph the curve and tangent line together. The tangent line intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.
- T c.** Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent line simultaneously.
- 68. a.** Find an equation for the line that is tangent to the curve $y = x^3 - 6x^2 + 5x$ at the origin.
- T b.** Graph the curve and tangent line together. The tangent line intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.
- T c.** Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent line simultaneously.

Theory and Examples

For Exercises 69 and 70, evaluate each limit by first converting each to a derivative at a particular x -value.

69. $\lim_{x \rightarrow 1} \frac{x^{50} - 1}{x - 1}$

70. $\lim_{x \rightarrow -1} \frac{x^{2/9} - 1}{x + 1}$

- 71.** Find the value of a that makes the following function differentiable for all x -values.

$$g(x) = \begin{cases} ax, & \text{if } x < 0 \\ x^2 - 3x, & \text{if } x \geq 0 \end{cases}$$

- 72.** Find the values of a and b that make the following function differentiable for all x -values.

$$f(x) = \begin{cases} ax + b, & x > -1 \\ bx^2 - 3, & x \leq -1 \end{cases}$$

- 73.** The general polynomial of degree n has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_n \neq 0$. Find $P'(x)$.

- 74. The body's reaction to medicine** The reaction of the body to a dose of medicine can sometimes be represented by an equation of the form

$$R = M^2 \left(\frac{C}{2} - \frac{M}{3} \right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM . This derivative, as a function of M , is called the sensitivity of the body to the medicine. In Section 4.6, we will see how to find the amount of medicine to which the body is most sensitive.

- 75.** Suppose that the function v in the Derivative Product Rule has a constant value c . What does the Derivative Product Rule then say? What does this say about the Derivative Constant Multiple Rule?

76. The Reciprocal Rule

- a.** The *Reciprocal Rule* says that at any point where the function $v(x)$ is differentiable and different from zero,

$$\frac{d}{dx} \left(\frac{1}{v} \right) = -\frac{1}{v^2} \frac{dv}{dx}.$$

Show that the Reciprocal Rule is a special case of the Derivative Quotient Rule.

- b.** Show that the Reciprocal Rule and the Derivative Product Rule together imply the Derivative Quotient Rule.

- 77. Generalizing the Product Rule** The Derivative Product Rule gives the formula

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + \frac{du}{dx} v$$

for the derivative of the product uv of two differentiable functions of x .

- a.** What is the analogous formula for the derivative of the product uvw of *three* differentiable functions of x ?
- b.** What is the formula for the derivative of the product $u_1 u_2 u_3 u_4$ of *four* differentiable functions of x ?
- c.** What is the formula for the derivative of a product $u_1 u_2 u_3 \cdots u_n$ of a finite number n of differentiable functions of x ?

- 78. Power Rule for negative integers** Use the Derivative Quotient Rule to prove the Power Rule for negative integers, that is,

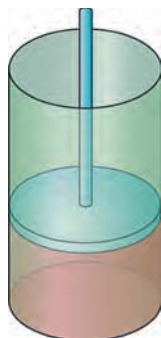
$$\frac{d}{dx}(x^{-m}) = -mx^{-m-1}$$

where m is a positive integer.

- 79. Cylinder pressure** If gas in a cylinder is maintained at a constant temperature T , the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which a , b , n , and R are constants. Find dP/dV . (See accompanying figure.)



- 80. The best quantity to order** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, TVs, brooms, or whatever the item might be); k is the cost of placing an order (the same, no matter how often you order); c is the cost of one item (a constant); m is the number of items sold each week (a constant); and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Find dA/dq and d^2A/dq^2 .

3.4 The Derivative as a Rate of Change

In this section we study applications where derivatives model the rates at which things change. It is natural to think of a quantity changing with respect to time, but other variables can be treated in the same way. For example, an economist may want to study how the cost of producing steel varies with the number of tons produced, or an engineer may want to know how the power output of a generator varies with its temperature.

Instantaneous Rates of Change

If we interpret the difference quotient $(f(x+h) - f(x))/h$ as the average rate of change in f over the interval from x to $x+h$, we can interpret its limit as $h \rightarrow 0$ as the instantaneous rate at which f is changing at the point x . This gives an important interpretation of the derivative.

DEFINITION The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Thus, instantaneous rates are limits of average rates.

It is conventional to use the word *instantaneous* even when x does not represent time. The word is, however, frequently omitted. When we say *rate of change*, we mean *instantaneous rate of change*.

EXAMPLE 1 The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4}D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?

Solution The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}.$$

When $D = 10$ m, the area is changing with respect to the diameter at the rate of $(\pi/2)10 = 5\pi \text{ m}^2/\text{m} \approx 15.71 \text{ m}^2/\text{m}$. ■

Motion Along a Line: Displacement, Velocity, Speed, Acceleration, and Jerk

Suppose that an object (or body, considered as a whole mass) is moving along a coordinate line (an s -axis), usually horizontal or vertical, so that we know its position s on that line as a function of time t :

$$s = f(t).$$

The **displacement** of the object over the time interval from t to $t + \Delta t$ (Figure 3.15) is

$$\Delta s = f(t + \Delta t) - f(t),$$

and the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the body's velocity at the exact instant t , we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt approaches zero. This limit is the derivative of f with respect to t .

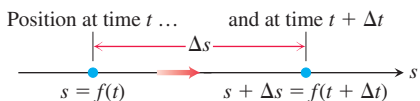
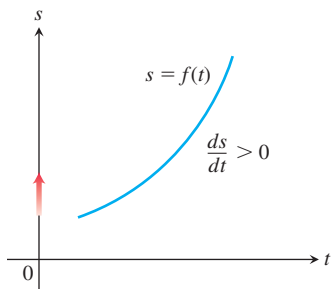
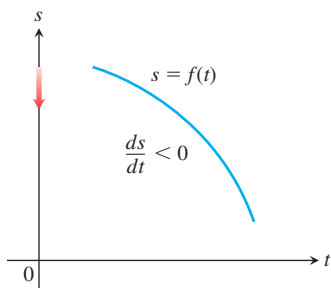


FIGURE 3.15 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$. Here the coordinate line is horizontal.



(a) s increasing:
positive slope so
moving upward



(b) s decreasing:
negative slope so
moving downward

FIGURE 3.16 For motion $s = f(t)$ along a straight line (the vertical axis), $v = ds/dt$ is (a) positive when s increases and (b) negative when s decreases.

DEFINITION Velocity (instantaneous velocity) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Besides telling how fast an object is moving along the horizontal line in Figure 3.15, its velocity tells the direction of motion. When the object is moving forward (s increasing), the velocity is positive; when the object is moving backward (s decreasing), the velocity is negative. If the coordinate line is vertical, the object moves upward for positive velocity and downward for negative velocity. The blue curves in Figure 3.16 represent position along the line over time; they do not portray the path of motion, which lies along the vertical s -axis.

If we drive to a friend's house and back at 50 km/h, say, the speedometer will show 50 on the way over but it will not show -50 on the way back, even though our distance from home is decreasing. The speedometer always shows *speed*, which is the absolute value of velocity. Speed measures the rate of progress regardless of direction.

DEFINITION Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left| \frac{ds}{dt} \right|$$

EXAMPLE 2 Figure 3.17 shows the graph of the velocity $v = f'(t)$ of a particle moving along a horizontal line as in Figure 3.15 (as opposed to the graph of a position function $s = f(t)$, such as in Figure 3.16). In the graph of the velocity function, it's not the

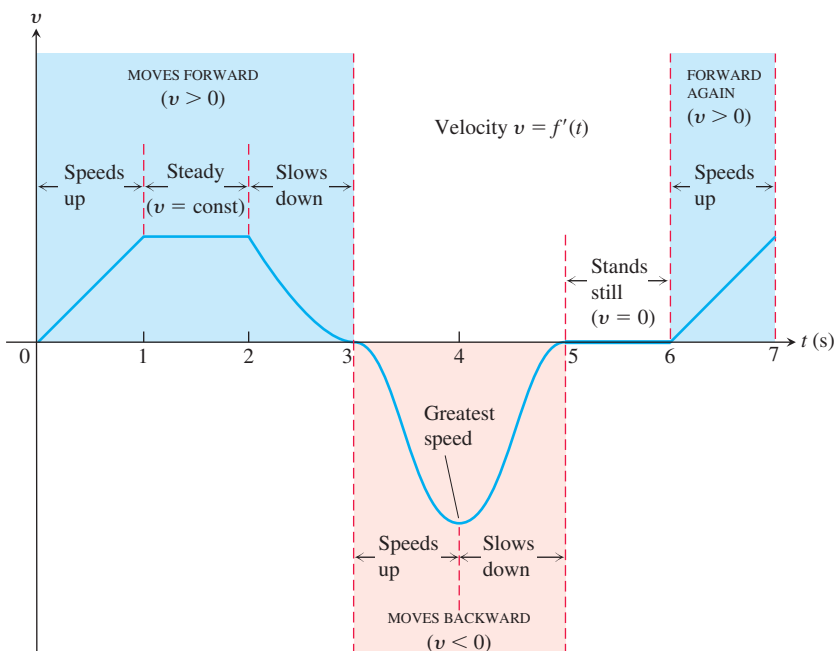


FIGURE 3.17 The velocity graph of a particle moving along a horizontal line, discussed in Example 2.

slope of the curve that tells us whether the particle is moving forward or backward along the line (which is not shown in the figure), but rather the sign of the velocity. Figure 3.17 shows that the particle moves forward for the first 3 s (when the velocity is positive), moves backward for the next 2 s (the velocity is negative), stands motionless for a full second, and then moves forward again. The particle is speeding up when its positive velocity increases during the first second, moves at a steady speed during the next second, and then slows down as the velocity decreases to zero during the third second. It stops for an instant at $t = 3$ s (when the velocity is zero) and reverses direction as the velocity starts to become negative. The particle is now moving backward and gaining in speed until $t = 4$ s, at which time it achieves its greatest speed during its backward motion. Continuing its backward motion at time $t = 4$, the particle starts to slow down again until it finally stops at time $t = 5$ (when the velocity is once again zero). The particle now remains motionless for one full second, and then moves forward again at $t = 6$ s, speeding up during the final second of the forward motion indicated in the velocity graph. ■

HISTORICAL BIOGRAPHY

Bernard Bolzano
(1781–1848)

Bolzano was born in Prague, Czechoslovakia. He studied at the University of Prague, where he took courses in philosophy, physics, and mathematics.

To know more, visit the companion Website.

The rate at which a body's velocity changes is the body's *acceleration*. The acceleration measures how quickly the body picks up or loses speed. In Chapter 12 we will study motion in the plane and in space, where acceleration of an object may also lead to a change in direction.

A sudden change in acceleration is called a *jerk*. When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt.

DEFINITIONS **Acceleration** is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Near the surface of Earth, all bodies fall with the same constant acceleration. Galileo's experiments with free fall (see Section 2.1) lead to the equation

$$s = \frac{1}{2}gt^2,$$

where s is the distance fallen and g is the acceleration due to Earth's gravity. This equation holds in a vacuum, where there is no air resistance, and closely models the fall of dense, heavy objects, such as rocks or steel tools, for the first few seconds of their fall, before the effects of air resistance are significant.

The value of g in the equation $s = (1/2)gt^2$ depends on the units used to measure t and s . With t in seconds (the usual unit), the value of g determined by measurement at sea level is approximately 9.8 m/s^2 (meters per second squared) in metric units. (This gravitational constant depends on the distance from Earth's center of mass, and is slightly lower on top of Mt. Everest, for example.)

The jerk associated with the constant acceleration of gravity ($g = 9.8 \text{ m/s}^2$) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

An object does not exhibit jerkiness during free fall.

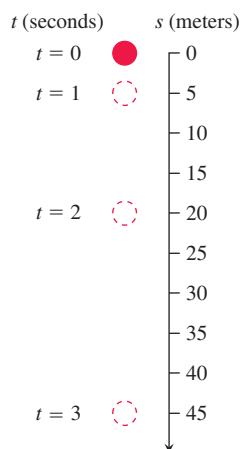


FIGURE 3.18 A ball bearing falling from rest (Example 3).

EXAMPLE 3 Figure 3.18 shows the free fall of a heavy ball bearing released from rest at time $t = 0 \text{ s}$.

- (a) How many meters does the ball fall in the first 3 s?
- (b) What are its velocity, speed, and acceleration when $t = 3$?

Solution

- (a) The metric free-fall equation is $s = 4.9t^2$. During the first 3 s, the ball falls

$$s(3) = 4.9(3)^2 = 44.1 \text{ m}.$$

- (b) At any time t , *velocity* is the derivative of position:

$$v(t) = s'(t) = \frac{d}{dt}(4.9t^2) = 9.8t.$$

At $t = 3$, the velocity is

$$v(3) = 29.4 \text{ m/s}$$

in the downward (increasing s) direction. The *speed* at $t = 3$ is

$$\text{speed} = |v(3)| = 29.4 \text{ m/s}.$$

The *acceleration* at any time t is

$$a(t) = v'(t) = s''(t) = 9.8 \text{ m/s}^2.$$

At $t = 3$, the acceleration is 9.8 m/s^2 . ■

EXAMPLE 4 A dynamite blast blows a heavy rock straight up with a launch velocity of 49 m/s (176.4 km/h) (Figure 3.19a). It reaches a height of $s = 49t - 4.9t^2 \text{ m}$ after t seconds.

- (a) How high does the rock go?
- (b) What are the velocity and speed of the rock when it is 78.4 m above the ground on the way up? On the way down?
- (c) What is the acceleration of the rock at any time t during its flight (after the blast)?
- (d) When does the rock hit the ground again?

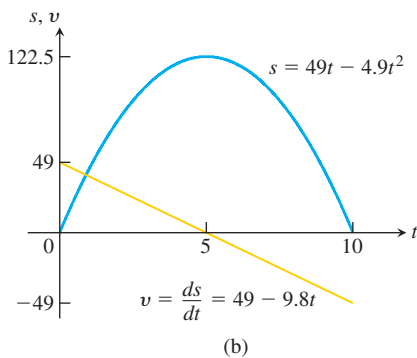
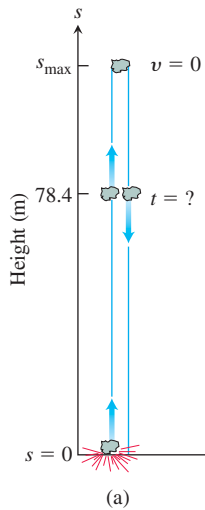


FIGURE 3.19 (a) The rock in Example 4. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

Solution

- (a) In the coordinate system we have chosen, s measures height from the ground up, so the velocity is positive on the way up and negative on the way down. The instant the rock is at its highest point is the one instant during the flight when the velocity is 0. To find the maximum height, all we need to do is to find when $v = 0$ and evaluate s at this time.

At any time t during the rock's motion, its velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(49t - 4.9t^2) = 49 - 9.8t \text{ m/s.}$$

The velocity is zero when

$$49 - 9.8t = 0 \quad \text{or} \quad t = 5 \text{ s.}$$

The rock's height at $t = 5$ s is

$$s_{\max} = s(5) = 49(5) - 4.9(5)^2 = 245 - 122.5 = 122.5 \text{ m.}$$

See Figure 3.19b.

- (b) To find the rock's velocity at 78.4 m on the way up and again on the way down, we first find the two values of t for which

$$s(t) = 49t - 4.9t^2 = 78.4.$$

To solve this equation, we write

$$\begin{aligned} 4.9t^2 - 49t + 78.4 &= 0 \\ 4.9(t^2 - 10t + 16) &= 0 \\ (t - 2)(t - 8) &= 0 \\ t &= 2 \text{ s, } t = 8 \text{ s.} \end{aligned}$$

The rock is 78.4 m above the ground 2 s after the explosion and again 8 s after the explosion. The rock's velocities at these times are

$$\begin{aligned} v(2) &= 49 - 9.8(2) = 49 - 19.6 = 29.4 \text{ m/s.} \\ v(8) &= 49 - 9.8(8) = 49 - 78.4 = -29.4 \text{ m/s.} \end{aligned}$$

At both instants, the rock's speed is 29.4 m/s. Since $v(2) > 0$, the rock is moving upward (s is increasing) at $t = 2$ s; it is moving downward (s is decreasing) at $t = 8$ s because $v(8) < 0$.

- (c) At any time during its flight following the explosion, the rock's acceleration is a constant

$$a = \frac{dv}{dt} = \frac{d}{dt}(49 - 9.8t) = -9.8 \text{ m/s}^2.$$

The acceleration is always downward and is the effect of gravity on the rock. As the rock rises, it slows down; as it falls, it speeds up.

- (d) The rock hits the ground at the positive time t for which $s = 0$. The equation $49t - 4.9t^2 = 0$ factors to give $4.9t(10 - t) = 0$, so it has solutions $t = 0$ and $t = 10$. At $t = 0$, the blast occurred and the rock was thrown upward. It returns to the ground 10 s later. ■

Derivatives in Economics and Biology

Economists have a specialized vocabulary for rates of change and derivatives. They call them *marginals*. In a manufacturing operation, the *cost of production* $c(x)$ is a function of x , the number of units produced. The **marginal cost of production** is the rate of change of cost with respect to level of production, so it is dc/dx .

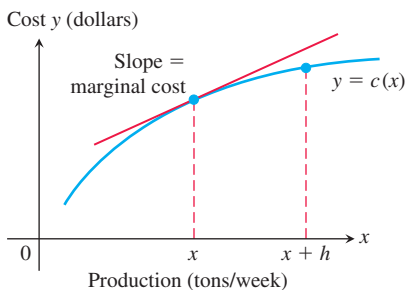


FIGURE 3.20 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x+h) - c(x)$.

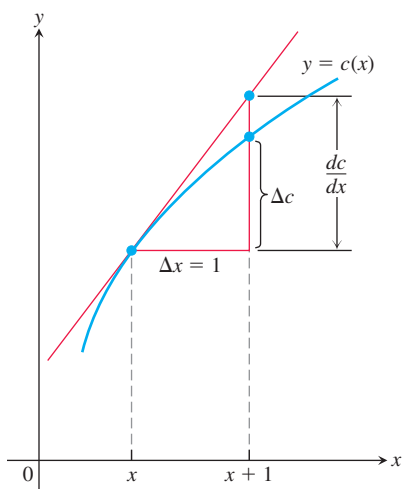


FIGURE 3.21 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

Suppose that $c(x)$ represents the dollars needed to produce x tons of steel in one week. It costs more to produce $x+h$ tons per week, and the cost difference, divided by h , is the average cost of producing each additional ton:

$$\frac{c(x+h) - c(x)}{h} = \text{average cost of each of the additional } h \text{ tons of steel produced.}$$

The limit of this ratio as $h \rightarrow 0$ is the *marginal cost* of producing more steel per week when the current weekly production is x tons (Figure 3.20):

$$\frac{dc}{dx} = \lim_{h \rightarrow 0} \frac{c(x+h) - c(x)}{h} = \text{marginal cost of production.}$$

Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one additional unit:

$$\frac{\Delta c}{\Delta x} = \frac{c(x+1) - c(x)}{1},$$

which is approximated by the value of dc/dx at x . This approximation is acceptable if the slope of the graph of c does not change quickly near x . Then the difference quotient will be close to its limit dc/dx , which is the rise in the tangent line if $\Delta x = 1$ (Figure 3.21). The approximation often works well for large values of x .

Economists often represent a total cost function by a cubic polynomial

$$c(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta,$$

where δ represents *fixed costs*, such as rent, heat, equipment capitalization, and management costs. The other terms represent *variable costs*, such as the costs of raw materials, taxes, and labor. Fixed costs are independent of the number of units produced, whereas variable costs depend on the quantity produced. A cubic polynomial is usually adequate to capture the cost behavior on a realistic quantity interval.

EXAMPLE 5 Suppose that it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 30 radiators are produced and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day, and what is your estimated increase in revenue and increase in profit for selling 11 radiators a day?

Solution The cost of producing one more radiator a day when 10 are produced is about $c'(10)$:

$$\begin{aligned} c'(x) &= \frac{d}{dx}(x^3 - 6x^2 + 15x) = 3x^2 - 12x + 15 \\ c'(10) &= 3(100) - 12(10) + 15 = 195. \end{aligned}$$

The additional cost will be about \$195. The marginal revenue is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12.$$

The marginal revenue function estimates the increase in revenue that will result from selling one additional unit. If you currently sell 10 radiators a day, you can expect your revenue to increase by about

$$r'(10) = 3(100) - 6(10) + 12 = \$252$$

if you increase sales to 11 radiators a day. The estimated increase in profit is obtained by subtracting the increased cost of \$195 from the increased revenue, leading to an estimated profit increase of $\$252 - \$195 = \$57$. ■

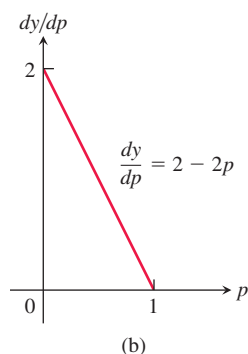
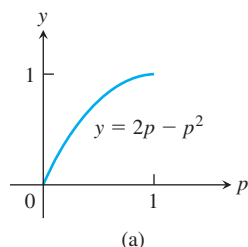


FIGURE 3.22 (a) The graph of $y = 2p - p^2$, describing the proportion of smooth-skin peas in the next generation. (b) The graph of dy/dp (Example 7).

EXAMPLE 6 Marginal rates frequently arise in discussions of tax rates. If your marginal income tax rate is 28% and your income increases by \$1000, you can expect to pay an extra \$280 in taxes. This does not mean that you pay 28% of your entire income in taxes. It just means that at your current income level I , the rate of increase of taxes T with respect to income is $dT/dI = 0.28$. You will pay \$0.28 in taxes out of every extra dollar you earn. As your income increases, you may land in a higher tax bracket, and your marginal rate will increase. ■

Sensitivity to Change

When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is sensitive to changes in x . The derivative $f'(x)$ is a measure of this sensitivity. The function is more sensitive when $|f'(x)|$ is larger (when the slope of the graph of f is steeper).

EXAMPLE 7 Genetic Data and Sensitivity to Change

The Austrian monk Gregor Johann Mendel (1822–1884), working with garden peas and other plants, provided the first scientific explanation of hybridization.

His careful records showed that if p (a number between 0 and 1) is the frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the frequency of the gene for wrinkled skin in peas, then the proportion of smooth-skin peas in the next generation will be

$$y = 2p(1 - p) + p^2 = 2p - p^2.$$

The graph of y versus p in Figure 3.22a suggests that the value of y is more sensitive to a change in p when p is small than when p is large. Indeed, this fact is borne out by the derivative graph in Figure 3.22b, which shows that dy/dp is close to 2 when p is near 0 and close to 0 when p is near 1.

The implication for genetics is that introducing a few more smooth skin genes into a population where the frequency of wrinkled-skin peas is large will have a more dramatic effect on later generations than will a similar increase when the population has a large proportion of smooth-skin peas. ■

EXERCISES 3.4

Motion Along a Coordinate Line

Exercises 1–6 give the positions $s = f(t)$ of a body moving on a coordinate line, with s in meters and t in seconds.

- Find the body's displacement and average velocity for the given time interval.
- Find the body's speed and acceleration at the endpoints of the interval.
- When, if ever, during the interval does the body change direction?

1. $s = t^2 - 3t + 2$, $0 \leq t \leq 2$

2. $s = 6t - t^2$, $0 \leq t \leq 6$

3. $s = -t^3 + 3t^2 - 3t$, $0 \leq t \leq 3$

4. $s = (t^4/4) - t^3 + t^2$, $0 \leq t \leq 3$

5. $s = \frac{25}{t^2} - \frac{5}{t}$, $1 \leq t \leq 5$ 6. $s = \frac{25}{t+5}$, $-4 \leq t \leq 0$

7. **Particle motion** At time t , the position of a body moving along the s -axis is $s = t^3 - 6t^2 + 9t$ m.

- Find the body's acceleration each time the velocity is zero.
- Find the body's speed each time the acceleration is zero.
- Find the total distance traveled by the body from $t = 0$ to $t = 2$.

8. **Particle motion** At time $t \geq 0$, the velocity of a body moving along the horizontal s -axis is $v = t^2 - 4t + 3$.

- Find the body's acceleration each time the velocity is zero.
- When is the body moving forward? Backward?
- When is the body's velocity increasing? Decreasing?

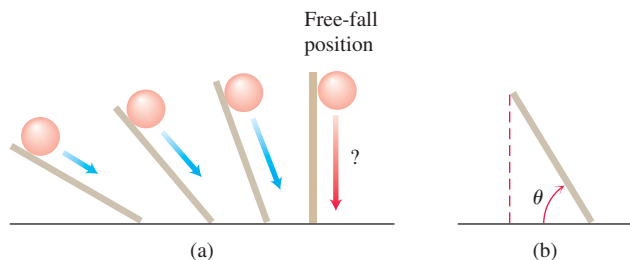
Free-Fall Applications

9. **Free fall on Mars and Jupiter** The equations for free fall at the surfaces of Mars and Jupiter (s in meters, t in seconds) are $s = 1.86t^2$ on Mars and $s = 11.44t^2$ on Jupiter. How long does it take a rock falling from rest to reach a velocity of 27.8 m/s (about 100 km/h) on each planet?

10. **Lunar projectile motion** A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/s (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ m in t s.

- Find the rock's velocity and acceleration at time t . (The acceleration in this case is the acceleration of gravity on the moon.)
- How long does it take the rock to reach its highest point?
- How high does the rock go?
- How long does it take the rock to reach half its maximum height?
- How long is the rock aloft?

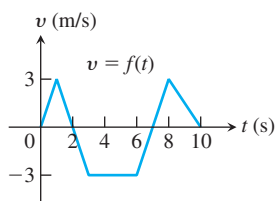
- 11. Finding g on a small airless planet** Explorers on a small airless planet used a spring gun to launch a ball bearing vertically upward from the surface at a launch velocity of 15 m/s. Because the acceleration of gravity at the planet's surface was g_s m/s², the explorers expected the ball bearing to reach a height of $s = 15t - (1/2)g_s t^2$ m t s later. The ball bearing reached its maximum height 20 s after being launched. What was the value of g_s ?
- 12. Speeding bullet** A 45-caliber bullet shot straight up from the surface of the moon would reach a height of $s = 250t - 0.8t^2$ m after t seconds. On Earth, in the absence of air, its height would be $s = 250t - 4.9t^2$ m after t seconds. How long will the bullet be aloft in each case? How high will the bullet go?
- 13. Free fall from the Tower of Pisa** Had Galileo dropped a cannonball from the Tower of Pisa, 56 m above the ground, the ball's height above the ground t seconds into the fall would have been $s = 56 - 4.9t^2$.
- What would have been the ball's velocity, speed, and acceleration at time t ?
 - About how long would it have taken the ball to hit the ground?
 - What would have been the ball's velocity at the moment of impact?
- 14. Galileo's free-fall formula** Galileo developed a formula for a body's velocity during free fall by rolling balls from rest down increasingly steep inclined planks and looking for a limiting formula that would predict a ball's behavior when the plank was vertical and the ball fell freely; see part (a) of the accompanying figure. He found that, for any given angle of the plank, the ball's velocity t seconds into motion was a constant multiple of t . That is, the velocity was given by a formula of the form $v = kt$. The value of the constant k depended on the inclination of the plank.
- In modern notation—part (b) of the figure—with distance in meters and time in seconds, what Galileo determined by experiment was that, for any given angle θ , the ball's velocity t s into the roll was $v = 9.8(\sin \theta)t$ m/s.



- What is the equation for the ball's velocity during free fall?
- Building on your work in part (a), what constant acceleration does a freely falling body experience near the surface of Earth?

Understanding Motion from Graphs

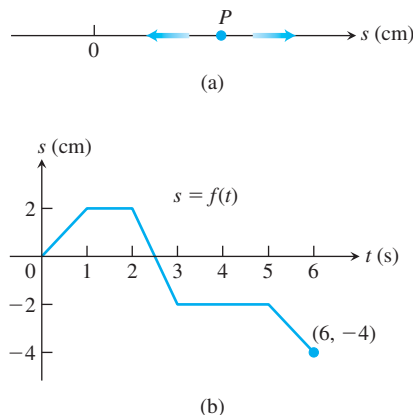
- 15.** The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/s) of a body moving along a coordinate line.



- When does the body reverse direction?

- When (approximately) is the body moving at a constant speed?
- Graph the body's speed for $0 \leq t \leq 10$.
- Graph the acceleration, where defined.

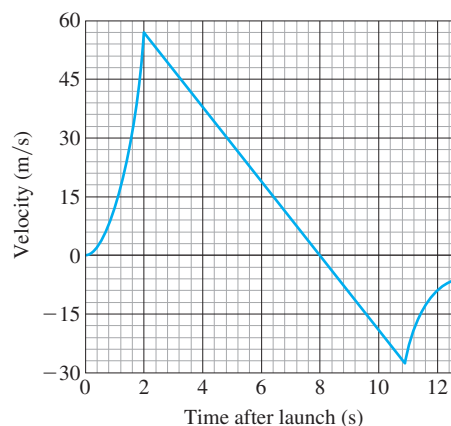
- 16.** A particle P moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of P as a function of time t .



- When is P moving to the left? Moving to the right? Standing still?
 - Graph the particle's velocity and speed (where defined).
- 17. Launching a rocket** When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts down. The parachute slows the rocket to keep it from breaking when it lands.

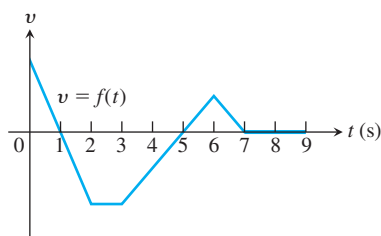
The figure here shows velocity data from the flight of the model rocket. Use the data to answer the following.

- How fast was the rocket climbing when the engine stopped?
- For how many seconds did the engine burn?

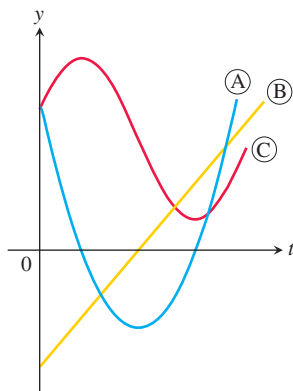


- When did the rocket reach its highest point? What was its velocity then?
- When did the parachute pop out? How fast was the rocket falling then?
- How long did the rocket fall before the parachute opened?
- When was the rocket's acceleration greatest?
- When was the acceleration constant? What was its value then (to the nearest integer)?

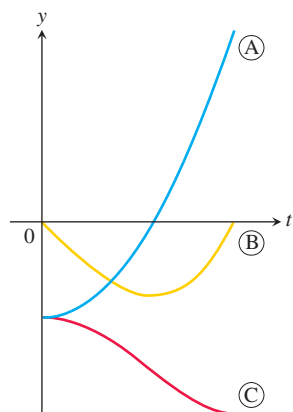
18. The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a horizontal coordinate line.



- When does the particle move forward? Move backward? Speed up? Slow down?
 - When is the particle's acceleration positive? Negative? Zero?
 - When does the particle move at its greatest speed?
 - When does the particle stand still for more than an instant?
19. The graphs in the accompanying figure show the position s , velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t . Which graph is which? Give reasons for your answers.



20. The graphs in the accompanying figure show the position s , the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t . Which graph is which? Give reasons for your answers.



Economics

21. **Marginal cost** Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.
- Find the average cost per machine of producing the first 100 washing machines.

- Find the marginal cost when 100 washing machines are produced.
- Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

22. **Marginal revenue** Suppose that the revenue from selling x washing machines is

$$r(x) = 20,000 \left(1 - \frac{1}{x} \right)$$

dollars.

- Find the marginal revenue when 100 machines are produced.
- Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 100 machines a week to 101 machines a week.
- Find the limit of $r'(x)$ as $x \rightarrow \infty$. How would you interpret this number?

Additional Applications

23. **Bacterium population** When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while, but then stopped growing and began to decline. The size of the population at time t (hours) was $b = 10^6 + 10^4 t - 10^3 t^2$. Find the growth rates at

- $t = 0$ hours.
- $t = 5$ hours.
- $t = 10$ hours.

24. **Body surface area** A typical male's body surface area S in square meters is often modeled by the formula $S = \frac{1}{60} \sqrt{wh}$, where h is the height in centimeters, and w the weight in kilograms, of the person. Find the rate of change of body surface area with respect to weight for males of constant height $h = 180$ cm. Does S increase more rapidly with respect to weight at lower or higher body weights? Explain.

- T** 25. **Draining a tank** It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

$$y = 6 \left(1 - \frac{t}{12} \right)^2 \text{ m.}$$

- Find the rate dy/dt (m/h) at which the tank is draining at time t .
 - When is the fluid level in the tank falling fastest? Slowest? What are the values of dy/dt at these times?
 - Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt .
26. **Draining a tank** The number of liters of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?
27. **Vehicular stopping distance** Based on data from the U.S. Bureau of Public Roads, a model for the total stopping distance of a moving car in terms of its speed is

$$s = 0.21v + 0.0063v^2,$$

where s is measured in meters and v is measured in km/h. The linear term $0.21v$ models the distance the car travels during the time the

driver perceives a need to stop until the brakes are applied, and the quadratic term $0.00636v^2$ models the additional braking distance once they are applied. Find ds/dv at $v = 50$ and $v = 100$ km/h, and interpret the meaning of the derivative.

28. Inflating a balloon The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

- At what rate (m^3/m) does the volume change with respect to the radius when $r = 2$ m?
- By approximately how much does the volume increase when the radius changes from 2 to 2.2 m?

29. Airplane takeoff Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where D is measured in meters from the starting point and t is measured in seconds from the time the brakes are released. The aircraft will become airborne when its speed reaches 200 km/h. How long will it take to become airborne, and what distance will it travel in that time?

30. Volcanic lava fountains Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 580 m straight into the air (a Hawaiian record). What was the lava's exit velocity in meters per second? In kilometers per hour? (*Hint:* If v_0

is the exit velocity of a particle of lava, its height t seconds later will be $s = v_0t - 4.9t^2$ m. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.)

Analyzing Motion Using Graphs

T Exercises 31–34 give the position function $s = f(t)$ of an object moving along the s -axis as a function of time t . Graph f together with the velocity function $v(t) = ds/dt = f'(t)$ and the acceleration function $a(t) = d^2s/dt^2 = f''(t)$. Comment on the object's behavior in relation to the signs and values of v and a . Include in your commentary such topics as the following:

- When is the object momentarily at rest?
- When does it move to the left (down) or to the right (up)?
- When does it change direction?
- When does it speed up and slow down?
- When is it moving fastest (highest speed)? Slowest?
- When is it farthest from the axis origin?

31. $s = 60t - 4.9t^2$, $0 \leq t \leq 12.5$ (a heavy object fired straight up from Earth's surface at 60 m/s)

32. $s = t^2 - 3t + 2$, $0 \leq t \leq 5$

33. $s = t^3 - 6t^2 + 7t$, $0 \leq t \leq 4$

34. $s = 4 - 7t + 6t^2 - t^3$, $0 \leq t \leq 4$

3.5 Derivatives of Trigonometric Functions

Many phenomena of nature are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits in Example 5a and Theorem 6 in Section 2.4 with the angle sum identity for the sine function (see Figure 1.46):

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition} \\ &= \lim_{h \rightarrow 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} && \text{Identity for } \sin(x+h) \\ &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \rightarrow 0} \left(\cos x \cdot \frac{\sin h}{h} \right) \\ &= \sin x \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\cos h - 1}{h}}_{\text{limit 0}} + \cos x \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\sin h}{h}}_{\text{limit 1}} && \text{Example 5a and Theorem 6, Section 2.4} \\ &= \sin x \cdot 0 + \cos x \cdot 1 = \cos x. \end{aligned}$$

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 We find derivatives of a difference, a product, and a quotient, each of which involves the sine function.

$$\begin{aligned} \text{(a)} \quad y = x^2 - \sin x: \quad \frac{dy}{dx} &= 2x - \frac{d}{dx}(\sin x) && \text{Difference Rule} \\ &= 2x - \cos x \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad y = e^x \sin x: \quad \frac{dy}{dx} &= e^x \frac{d}{dx}(\sin x) + \left(\frac{d}{dx}e^x\right)\sin x && \text{Product Rule} \\ &= e^x \cos x + e^x \sin x \\ &= e^x(\cos x + \sin x) \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad y = \frac{\sin x}{x}: \quad \frac{dy}{dx} &= \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} && \text{Quotient Rule} \\ &= \frac{x \cos x - \sin x}{x^2} \end{aligned}$$

Derivative of the Cosine Function

With the help of the angle sum formula for the cosine function (see Figure 1.46),

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we can compute the limit of the difference quotient:

$$\begin{aligned} \frac{d}{dx}(\cos x) &= \lim_{h \rightarrow 0} \frac{\cos(x + h) - \cos x}{h} && \text{Derivative definition} \\ &= \lim_{h \rightarrow 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} && \text{Cosine angle sum identity} \\ &= \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \left(\cos x \cdot \frac{\cos h - 1}{h} \right) - \lim_{h \rightarrow 0} \left(\sin x \cdot \frac{\sin h}{h} \right) \\ &= \cos x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} && \text{Example 5a and Theorem 6, Section 2.4} \\ &= \cos x \cdot 0 - \sin x \cdot 1 \\ &= -\sin x. \end{aligned}$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

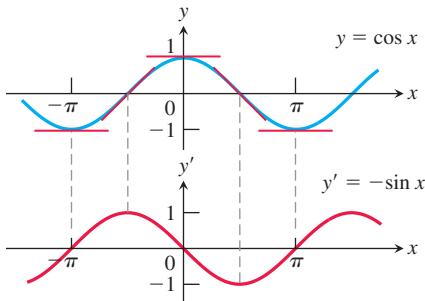


FIGURE 3.23 The curve $y' = -\sin x$ as the graph of the slopes of the tangent lines to the curve $y = \cos x$.

Figure 3.23 shows a way to visualize this result by graphing the slopes of the tangent lines to the curve $y = \cos x$.

EXAMPLE 2 We find derivatives of the cosine function in combinations with other functions.

$$\text{(a)} \quad y = 5e^x + \cos x:$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(5e^x) + \frac{d}{dx}(\cos x) && \text{Sum Rule} \\ &= 5e^x - \sin x \end{aligned}$$

(b) $y = \sin x \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \left(\frac{d}{dx}(\sin x)\right) \cos x && \text{Product Rule} \\ &= (\sin x)(-\sin x) + (\cos x)(\cos x) \\ &= \cos^2 x - \sin^2 x\end{aligned}$$

(c) $y = \frac{\cos x}{1 - \sin x}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1 - \sin x) \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} && \text{Quotient Rule} \\ &= \frac{(1 - \sin x)(-\sin x) - (\cos x)(0 - \cos x)}{(1 - \sin x)^2} \\ &= \frac{1 - \sin x}{(1 - \sin x)^2} && \sin^2 x + \cos^2 x = 1 \\ &= \frac{1}{1 - \sin x}\end{aligned}$$

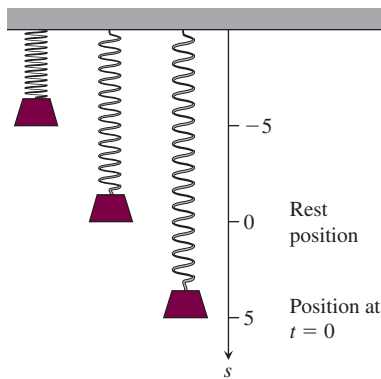


FIGURE 3.24 A weight hanging from a vertical spring and then displaced oscillates above and below its rest position (Example 3).

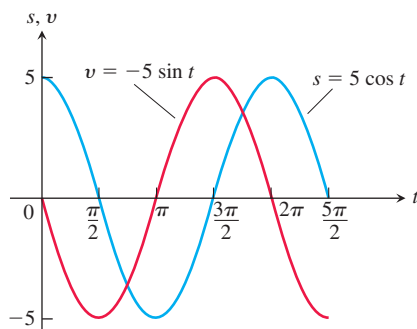


FIGURE 3.25 The graphs of the position and velocity of the weight in Example 3.

Simple Harmonic Motion

Simple harmonic motion models the motion of an object or weight bobbing freely up and down on the end of a spring, with no resistance. The motion is periodic and repeats indefinitely, so we represent it using trigonometric functions. The next example models motion with no opposing forces (such as friction).

EXAMPLE 3 A weight hanging from a spring (Figure 3.24) is stretched down 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time t ?

Solution We have

Position: $s = 5 \cos t$

Velocity: $v = \frac{ds}{dt} = \frac{d}{dt}(5 \cos t) = -5 \sin t$

Acceleration: $a = \frac{dv}{dt} = \frac{d}{dt}(-5 \sin t) = -5 \cos t.$

Notice how much we can learn from these equations:

1. As time passes, the weight moves down and up between $s = -5$ and $s = 5$ on the s -axis. The amplitude of the motion is 5. The period of the motion is 2π , the period of the cosine function.
2. The velocity $v = -5 \sin t$ attains its greatest magnitude, 5, when $\cos t = 0$, as the graphs in Figure 3.25 show. Hence, the speed of the weight, $|v| = 5|\sin t|$, is greatest when $\cos t = 0$, that is, when $s = 0$ (the rest position). The speed of the weight is zero when $\sin t = 0$. This occurs when $s = 5 \cos t = \pm 5$, at the endpoints of the interval of motion. At these points the weight reverses direction.
3. The weight is acted on by the spring and by gravity. When the weight is below the rest position, the combined forces pull it up, and when it is above the rest position, they pull it down. The weight's acceleration is always proportional to the negative of its displacement. This property of springs is called *Hooke's Law*, and is studied further in Section 6.5.

4. The acceleration, $a = -5 \cos t$, is zero only at the rest position, where $\cos t = 0$ and the force of gravity and the force from the spring balance each other. When the weight is anywhere else, the two forces are unequal, and acceleration is nonzero. The acceleration is greatest in magnitude at the points farthest from the rest position, where $\cos t = \pm 1$. ■

EXAMPLE 4 The jerk associated with the simple harmonic motion in Example 3 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5 \cos t) = 5 \sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the rest position, where the acceleration changes direction and sign. ■

Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x , it follows from the Quotient Rule that the related functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives are given by the following formulas. Notice the negative signs in the derivative formulas for the cofunctions.

The derivatives of the other trigonometric functions:

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \sec^2 x & \frac{d}{dx}(\cot x) &= -\csc^2 x \\ \frac{d}{dx}(\sec x) &= \sec x \tan x & \frac{d}{dx}(\csc x) &= -\csc x \cot x \end{aligned}$$

To show a typical calculation, we find the derivative of the tangent function. The other derivations are left for Exercise 54.

EXAMPLE 5 Find $d(\tan x)/dx$.

Solution We use the Derivative Quotient Rule to calculate the derivative:

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} && \text{Quotient Rule} \\ &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x. \end{aligned}$$

EXAMPLE 6 Find y'' if $y = \sec x$.**Solution** Finding the second derivative involves a combination of trigonometric derivatives.

$$y = \sec x$$

$$y' = \sec x \tan x$$

Derivative rule for secant function

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x)$$

Derivative Product Rule

$$= (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x)$$

Derivative rules

$$= \sec^3 x + \sec x \tan^2 x$$

EXERCISES 3.5**Derivatives**In Exercises 1–18, find dy/dx .

1. $y = -10x + 3 \cos x$

2. $y = \frac{3}{x} + 5 \sin x$

3. $y = x^2 \cos x$

4. $y = \sqrt{x} \sec x + 3$

5. $y = \csc x - 4\sqrt{x} + \frac{7}{e^x}$

6. $y = x^2 \cot x - \frac{1}{x^2}$

7. $f(x) = \sin x \tan x$

8. $g(x) = \frac{\cos x}{\sin^2 x}$

9. $y = xe^{-x} \sec x$

10. $y = (\sin x + \cos x) \sec x$

11. $y = \frac{\cot x}{1 + \cot x}$

12. $y = \frac{\cos x}{1 + \sin x}$

13. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

14. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$

15. $y = (\sec x + \tan x)(\sec x - \tan x)$

16. $y = x^2 \cos x - 2x \sin x - 2 \cos x$

17. $f(x) = x^3 \sin x \cos x$

18. $g(x) = (2 - x) \tan^2 x$

In Exercises 19–22, find ds/dt .

19. $s = \tan t - e^{-t}$

20. $s = t^2 - \sec t + 5e^t$

21. $s = \frac{1 + \csc t}{1 - \csc t}$

22. $s = \frac{\sin t}{1 - \cos t}$

In Exercises 23–26, find $dr/d\theta$.

23. $r = 4 - \theta^2 \sin \theta$

24. $r = \theta \sin \theta + \cos \theta$

25. $r = \sec \theta \csc \theta$

26. $r = (1 + \sec \theta) \sin \theta$

In Exercises 27–32, find dp/dq .

27. $p = 5 + \frac{1}{\cot q}$

28. $p = (1 + \csc q) \cos q$

29. $p = \frac{\sin q + \cos q}{\cos q}$

30. $p = \frac{\tan q}{1 + \tan q}$

31. $p = \frac{q \sin q}{q^2 - 1}$

32. $p = \frac{3q + \tan q}{q \sec q}$

33. Find y'' if

a. $y = \csc x$.

b. $y = \sec x$.

34. Find $y^{(4)} = d^4 y/dx^4$ if

a. $y = -2 \sin x$.

b. $y = 9 \cos x$.

Tangent LinesIn Exercises 35–38, graph the curves over the given intervals, together with their tangent lines at the given values of x . Label each curve and tangent line with its equation.

35. $y = \sin x, -3\pi/2 \leq x \leq 2\pi$
 $x = -\pi, 0, 3\pi/2$

36. $y = \tan x, -\pi/2 < x < \pi/2$
 $x = -\pi/3, 0, \pi/3$

37. $y = \sec x, -\pi/2 < x < \pi/2$
 $x = -\pi/3, \pi/4$

38. $y = 1 + \cos x, -3\pi/2 \leq x \leq 2\pi$
 $x = -\pi/3, 3\pi/2$

T Do the graphs of the functions in Exercises 39–44 have any horizontal tangent lines in the interval $0 \leq x \leq 2\pi$? If so, where? If not, why not? Visualize your findings by graphing the functions with a grapher.

39. $y = x + \sin x$

40. $y = 2x + \sin x$

41. $y = x - \cot x$

42. $y = x + 2 \cos x$

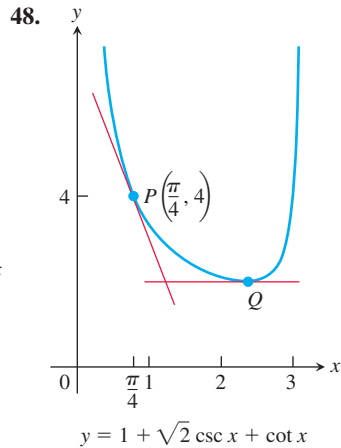
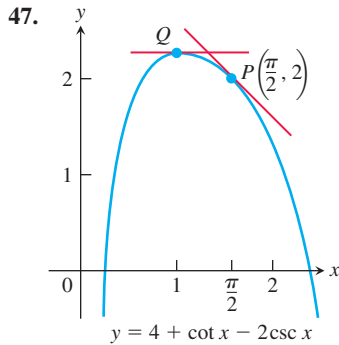
43. $y = \frac{\sec x}{3 + \sec x}$

44. $y = \frac{\cos x}{3 - 4 \sin x}$

45. Find all points on the curve $y = \tan x, -\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent lines together, labeling each with its equation.

46. Find all points on the curve $y = \cot x$, $0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent lines together, labeling each with its equation.

In Exercises 47 and 48, find an equation for (a) the tangent line to the curve at P and (b) the horizontal tangent line to the curve at Q .



Theory and Examples

The equations in Exercises 49 and 50 give the position $s = f(t)$ of a body moving on a coordinate line (s in meters, t in seconds). Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ s.

49. $s = 2 - 2 \sin t$

50. $s = \sin t + \cos t$

51. Is there a value of c that will make

$$f(x) = \begin{cases} \frac{\sin^2 3x}{x^2}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

52. Is there a value of b that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$

continuous at $x = 0$? Differentiable at $x = 0$? Give reasons for your answers.

53. By computing the first few derivatives and looking for a pattern, find the following derivatives.

a. $\frac{d^{999}}{dx^{999}}(\cos x)$

b. $\frac{d^{110}}{dx^{110}}(\sin x - 3 \cos x)$

c. $\frac{d^{73}}{dx^{73}}(x \sin x)$

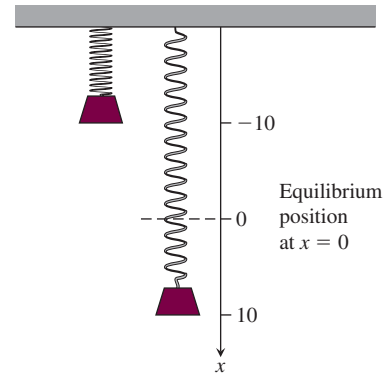
54. Derive the formula for the derivative with respect to x of

a. $\sec x$. b. $\csc x$. c. $\cot x$.

55. A weight is attached to a spring and reaches its equilibrium position ($x = 0$). It is then set in motion resulting in a displacement of

$$x = 10 \cos t,$$

where x is measured in centimeters and t is measured in seconds. See the accompanying figure.



- a. Find the spring's displacement when $t = 0$, $t = \pi/3$, and $t = 3\pi/4$.
b. Find the spring's velocity when $t = 0$, $t = \pi/3$, and $t = 3\pi/4$.

56. Assume that a particle's position on the x -axis is given by

$$x = 3 \cos t + 4 \sin t,$$

where x is measured in meters and t is measured in seconds.

- a. Find the particle's position when $t = 0$, $t = \pi/2$, and $t = \pi$.
b. Find the particle's velocity when $t = 0$, $t = \pi/2$, and $t = \pi$.

- T** 57. Graph $y = \cos x$ for $-\pi \leq x \leq 2\pi$. On the same screen, graph

$$y = \frac{\sin(x+h) - \sin x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1 . Then, in a new window, try $h = -1, -0.5$, and -0.3 . What happens as $h \rightarrow 0^+$? As $h \rightarrow 0^-$? What phenomenon is being illustrated here?

- T** 58. Graph $y = -\sin x$ for $-\pi \leq x \leq 2\pi$. On the same screen, graph

$$y = \frac{\cos(x+h) - \cos x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1 . Then, in a new window, try $h = -1, -0.5$, and -0.3 . What happens as $h \rightarrow 0^+$? As $h \rightarrow 0^-$? What phenomenon is being illustrated here?

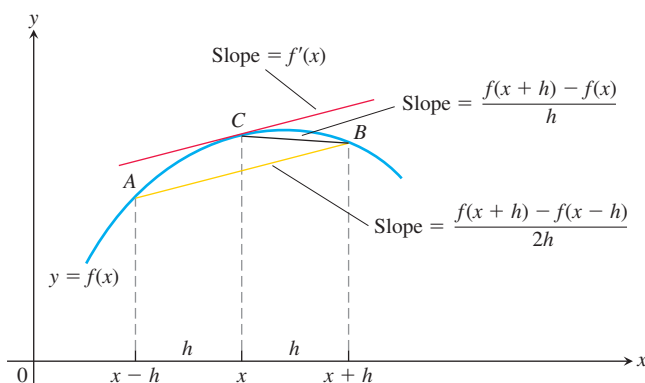
- T** 59. **Centered difference quotients** The centered difference quotient

$$\frac{f(x+h) - f(x-h)}{2h}$$

is used to approximate $f'(x)$ in numerical work because (1) its limit as $h \rightarrow 0$ equals $f'(x)$ when $f'(x)$ exists, and (2) it usually gives a better approximation of $f'(x)$ for a given value of h than the difference quotient

$$\frac{f(x+h) - f(x)}{h}.$$

See the accompanying figure.



- a. To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with

$$y = \frac{\sin(x+h) - \sin(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3 . Compare the results with those obtained in Exercise 57 for the same values of h .

- b. To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with

$$y = \frac{\cos(x+h) - \cos(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3 . Compare the results with those obtained in Exercise 58 for the same values of h .

60. **A caution about centered difference quotients** (Continuation of Exercise 59.) The quotient

$$\frac{f(x+h) - f(x-h)}{2h}$$

may have a limit as $h \rightarrow 0$ when f has no derivative at x . As a case in point, take $f(x) = |x|$ and calculate

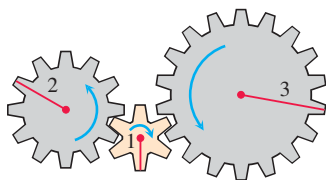
$$\lim_{h \rightarrow 0} \frac{|0+h| - |0-h|}{2h}.$$

As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$. *Moral:* Before using a centered difference quotient, be sure the derivative exists.

61. **Slopes on the graph of the tangent function** Graph $y = \tan x$ and its derivative together on $(-\pi/2, \pi/2)$. Does the graph of the tangent function appear to have a smallest slope? A largest slope? Is the slope ever negative? Give reasons for your answers.

62. **Exploring $(\sin kx)/x$** Graph $y = (\sin x)/x$, $y = (\sin 2x)/x$, and $y = (\sin 4x)/x$ together over the interval $-2 \leq x \leq 2$. Where does each graph appear to cross the y -axis? Do the graphs really intersect the axis? What would you expect the graphs of $y = (\sin 5x)/x$ and $y = (\sin(-3x))/x$ to do as $x \rightarrow 0$? Why? What about the graph of $y = (\sin kx)/x$ for other values of k ? Give reasons for your answers.

3.6 The Chain Rule



C: y turns B: u turns A: x turns

FIGURE 3.26 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

How do we differentiate $F(x) = \sin(x^2 - 4)$? This function is the composition $f \circ g$ of two functions $y = f(u) = \sin u$ and $u = g(x) = x^2 - 4$ that we know how to differentiate. The answer, given by the *Chain Rule*, says that the derivative is the product of the derivatives of f and g . We develop the rule in this section.

Derivative of a Composite Function

The function $y = \frac{3}{2}x = \frac{1}{2}(3x)$ is the composition of the functions $y = \frac{1}{2}u$ and $u = 3x$.

We have

$$\frac{dy}{dx} = \frac{3}{2}, \quad \frac{dy}{du} = \frac{1}{2}, \quad \text{and} \quad \frac{du}{dx} = 3.$$

Since $\frac{3}{2} = \frac{1}{2} \cdot 3$, we see in this case that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

If we think of the derivative as a rate of change, this relationship is intuitively reasonable. If $y = f(u)$ changes half as fast as u , and $u = g(x)$ changes three times as fast as x , then we expect y to change $3/2$ times as fast as x . This effect is much like that of a multiple gear train (Figure 3.26). Let's look at another example.

EXAMPLE 1 The function

$$y = (3x^2 + 1)^2$$

is obtained by composing the functions $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned} \frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x && \text{Substitute for } u. \\ &= 36x^3 + 12x. \end{aligned}$$

Calculating the derivative from the expanded formula $(3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$ gives the same result:

$$\frac{dy}{dx} = \frac{d}{dx}(9x^4 + 6x^2 + 1) = 36x^3 + 12x. \quad \blacksquare$$

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x . This is known as the Chain Rule (Figure 3.27).

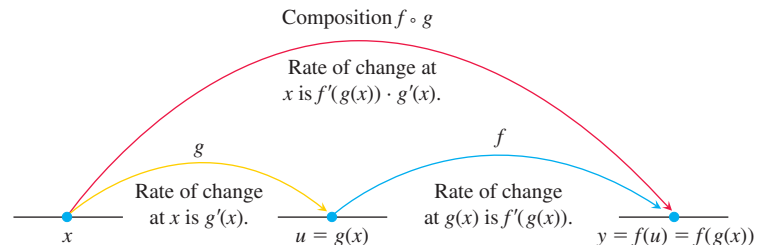


FIGURE 3.27 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x .

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x , then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x , and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

A Proof of One Case of the Chain Rule: Let Δu be the change in u when x changes by Δx , so that

$$\Delta u = g(x + \Delta x) - g(x).$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u).$$

If $\Delta u \neq 0$, we can write the fraction $\Delta y/\Delta x$ as the product

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \quad (1)$$

and take the limit as $\Delta x \rightarrow 0$:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \quad (\text{Note that } \Delta u \rightarrow 0 \text{ as } \Delta x \rightarrow 0 \text{ since } g \text{ is continuous.}) \\ &= \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \\ &= \frac{dy}{du} \cdot \frac{du}{dx}. \end{aligned}$$

The problem with this argument is that if the function $g(x)$ oscillates rapidly near x , then Δu can be zero even when $\Delta x \neq 0$, so the cancelation of Δu in Equation (1) would be invalid. A complete proof requires a different approach that avoids this problem, and we give one such proof in Section 3.11. ■

EXAMPLE 2 An object moves along the x -axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t .

Solution We know that the velocity is dx/dt . In this instance, x is a composition of two functions: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\begin{aligned} \frac{dx}{du} &= -\sin(u) & x &= \cos(u) \\ \frac{du}{dt} &= 2t. & u &= t^2 + 1 \end{aligned}$$

By the Chain Rule,

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\ &= -\sin(u) \cdot 2t \\ &= -\sin(t^2 + 1) \cdot 2t \\ &= -2t \sin(t^2 + 1). \end{aligned}$$

Ways to Write the Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}$$

“Outside-Inside” Rule

A difficulty with the Leibniz notation is that it doesn’t state specifically where the derivatives in the Chain Rule are supposed to be evaluated. So it sometimes helps to write the Chain Rule using functional notation. If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the “outside” function f and evaluate this derivative at the “inside” function $g(x)$ left alone; then multiply by the derivative of the “inside function.”

EXAMPLE 3 Differentiate $\sin(x^2 + e^x)$ with respect to x .

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx} \sin(\underbrace{x^2 + e^x}_{\text{inside}}) = \cos(\underbrace{x^2 + e^x}_{\text{inside left alone}}) \cdot (\underbrace{2x + e^x}_{\text{derivative of the inside}}).$$

EXAMPLE 4 Differentiate $y = e^{\cos x}$.

Solution Here the inside function is $u = g(x) = \cos x$ and the outside function is the exponential function $f(x) = e^x$. Applying the Chain Rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\cos x}) = e^{\cos x} \frac{d}{dx}(\cos x) = e^{\cos x}(-\sin x) = -e^{\cos x} \sin x.$$

Generalizing Example 4, we see that the Chain Rule gives the formula

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}.$$

For example,

$$\frac{d}{dx}(e^{kx}) = e^{kx} \cdot \frac{d}{dx}(kx) = ke^{kx}, \quad \text{for any constant } k$$

and

$$\frac{d}{dx}(e^{x^2}) = e^{x^2} \cdot \frac{d}{dx}(x^2) = 2xe^{x^2}.$$

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative.

EXAMPLE 5 Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t . Therefore, by the Chain Rule,

$$\begin{aligned} g'(t) &= \frac{d}{dt} \tan(5 - \sin 2t) \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) && \text{Derivative of } -\sin u \text{ with } u = 2t \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\ &= -2(\cos 2t)\sec^2(5 - \sin 2t). \end{aligned}$$

The Chain Rule with Powers of a Function

If n is any real number and f is a power function, $f(u) = u^n$, the Power Rule tells us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x , then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}. \quad \frac{d}{du}(u^n) = nu^{n-1}$$

HISTORICAL BIOGRAPHY

Johann Bernoulli
(1667–1748)

Johann Bernoulli was born in Switzerland and attended the University of Basel. His doctoral dissertation was in mathematics despite its medical title, which was used to hide his mathematical work from his father who wanted Johann to become a doctor.

To know more, visit the companion Website.

EXAMPLE 6 The Power Chain Rule simplifies computing the derivative of a power of an expression.

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4) && \text{Power Chain Rule with } u = 5x^3 - x^4, n = 7 \\ &= 7(5x^3 - x^4)^6(15x^2 - 4x^3) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}(3x-2)^{-1} && \text{Power Chain Rule with } u = 3x-2, n = -1 \\ &= -1(3x-2)^{-2} \frac{d}{dx}(3x-2) \\ &= -1(3x-2)^{-2}(3) \\ &= -\frac{3}{(3x-2)^2} \end{aligned}$$

In part (b) we could also find the derivative with the Quotient Rule.

$$\begin{aligned} \text{(c)} \quad \frac{d}{dx}(\sin^5 x) &= 5 \sin^4 x \cdot \frac{d}{dx} \sin x && \text{Power Chain Rule with } u = \sin x, n = 5, \\ &= 5 \sin^4 x \cos x && \text{because } \sin^n x \text{ means } (\sin x)^n, n \neq -1 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \frac{d}{dx}(e^{\sqrt{3x+1}}) &= e^{\sqrt{3x+1}} \cdot \frac{d}{dx}(\sqrt{3x+1}) \\ &= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x+1)^{-1/2} \cdot 3 && \text{Power Chain Rule with } u = 3x+1, n = 1/2 \\ &= \frac{3}{2\sqrt{3x+1}} e^{\sqrt{3x+1}} \end{aligned}$$

EXAMPLE 7 In Example 4 of Section 3.2 we saw that the absolute value function $y = |x|$ is not differentiable at $x = 0$. However, the function is differentiable at all other real numbers, as we now show. Since $|x| = \sqrt{x^2}$, we can derive the following formula, which gives an alternative to the more direct analysis seen before.

Derivative of the Absolute Value Function

$$\begin{aligned} \frac{d}{dx}(|x|) &= \frac{x}{|x|}, \quad x \neq 0 \\ &= \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}(|x|) &= \frac{d}{dx} \sqrt{x^2} \\ &= \frac{1}{2\sqrt{x^2}} \cdot \frac{d}{dx}(x^2) && \text{Power Chain Rule with } u = x^2, n = 1/2, x \neq 0 \\ &= \frac{1}{2|x|} \cdot 2x && \sqrt{x^2} = |x| \\ &= \frac{x}{|x|}, \quad x \neq 0. \end{aligned}$$

EXAMPLE 8 Show that the slope of every line tangent to the curve $y = 1/(1-2x)^3$ is positive.

Solution We find the derivative:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(1-2x)^{-3} \\ &= -3(1-2x)^{-4} \cdot \frac{d}{dx}(1-2x) && \text{Power Chain Rule with } u = (1-2x), n = -3 \\ &= -3(1-2x)^{-4} \cdot (-2) \\ &= \frac{6}{(1-2x)^4}. \end{aligned}$$

At any point (x, y) on the curve, the denominator is nonzero, and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1-2x)^4},$$

which is the quotient of two positive numbers.

EXAMPLE 9 The formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since $180^\circ = \pi$ radians, $x^\circ = \pi x/180$ radians, where x° is the size of the angle measured in degrees.

By the Chain Rule,

$$\frac{d}{dx} \sin(x^\circ) = \frac{d}{dx} \sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180} \cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180} \cos(x^\circ).$$

See Figure 3.28. Similarly, the derivative of $\cos(x^\circ)$ is $-(\pi/180) \sin(x^\circ)$.

The factor $\pi/180$ would propagate with repeated differentiation, showing an advantage for the use of radian measure in computations. ■

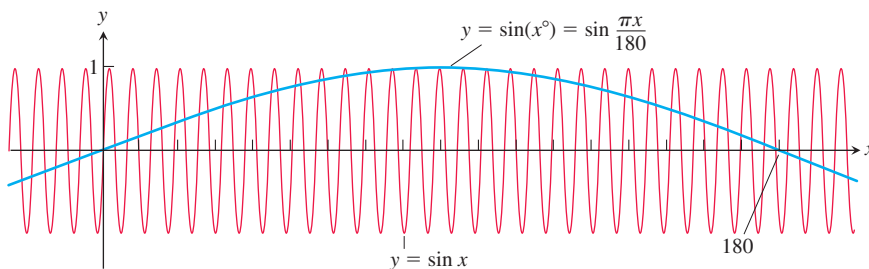


FIGURE 3.28 The function $\sin(x^\circ)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$ at $x = 0$ (Example 9).

EXERCISES 3.6

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9$, $u = (1/2)x^4$
2. $y = 2u^3$, $u = 8x - 1$
3. $y = \sin u$, $u = 3x + 1$
4. $y = \cos u$, $u = e^{-x}$
5. $y = \sqrt{u}$, $u = \sin x$
6. $y = \sin u$, $u = x - \cos x$
7. $y = \tan u$, $u = \pi x^2$
8. $y = -\sec u$, $u = \frac{1}{x} + 7x$

In Exercises 9–22, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x .

9. $y = (2x + 1)^5$
10. $y = (4 - 3x)^9$
11. $y = \left(1 - \frac{x}{7}\right)^{-7}$
12. $y = \left(\frac{\sqrt{x}}{2} - 1\right)^{-10}$
13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$
14. $y = \sqrt{3x^2 - 4x + 6}$
15. $y = \sec(\tan x)$
16. $y = \cot\left(\pi - \frac{1}{x}\right)$
17. $y = \tan^3 x$
18. $y = 5 \cos^{-4} x$
19. $y = e^{-5x}$
20. $y = e^{2x/3}$
21. $y = e^{5-7x}$
22. $y = e^{(4\sqrt{x}+x^2)}$

Find the derivatives of the functions in Exercises 23–50.

23. $p = \sqrt{3-t}$
24. $q = \sqrt[3]{2r-r^2}$
25. $s = \frac{4}{3\pi} \sin 3t + \frac{4}{5\pi} \cos 5t$
26. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$
27. $r = (\csc \theta + \cot \theta)^{-1}$
28. $r = 6(\sec \theta - \tan \theta)^{3/2}$
29. $y = x^2 \sin^4 x + x \cos^{-2} x$
30. $y = \frac{1}{x} \sin^{-5} x - \frac{x}{3} \cos^3 x$
31. $y = \frac{1}{18}(3x-2)^6 + \left(4 - \frac{1}{2x^2}\right)^{-1}$

32. $y = (5-2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4$
33. $y = (4x+3)^4(x+1)^{-3}$
34. $y = (2x-5)^{-1}(x^2-5x)^6$
35. $y = xe^{-x} + e^{x^3}$
36. $y = (1+2x)e^{-2x}$
37. $y = (x^2-2x+2)e^{5x/2}$
38. $y = (9x^2-6x+2)e^{x^3}$
39. $h(x) = x \tan(2\sqrt{x}) + 7$
40. $k(x) = x^2 \sec\left(\frac{1}{x}\right)$
41. $f(x) = \sqrt{7+x \sec x}$
42. $g(x) = \frac{\tan 3x}{(x+7)^4}$
43. $f(\theta) = \left(\frac{\sin \theta}{1+\cos \theta}\right)^2$
44. $g(t) = \left(\frac{1+\sin 3t}{3-2t}\right)^{-1}$
45. $r = \sin(\theta^2) \cos(2\theta)$
46. $r = \sec \sqrt{\theta} \tan\left(\frac{1}{\theta}\right)$
47. $q = \sin\left(\frac{t}{\sqrt{t+1}}\right)$
48. $q = \cot\left(\frac{\sin t}{t}\right)$
49. $y = \cos(e^{-\theta^2})$
50. $y = \theta^3 e^{-2\theta} \cos 5\theta$

In Exercises 51–70, find dy/dt .

51. $y = \sin^2(\pi t - 2)$
52. $y = \sec^2 \pi t$
53. $y = (1 + \cos 2t)^{-4}$
54. $y = (1 + \cot(t/2))^{-2}$
55. $y = (t \tan t)^{10}$
56. $y = (t^{-3/4} \sin t)^{4/3}$
57. $y = e^{\cos^2(\pi t - 1)}$
58. $y = (e^{\sin(t/2)})^3$
59. $y = \left(\frac{t^2}{t^3 - 4t}\right)^3$
60. $y = \left(\frac{3t-4}{5t+2}\right)^{-5}$
61. $y = \sin(\cos(2t-5))$
62. $y = \cos\left(5 \sin\left(\frac{t}{3}\right)\right)$
63. $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$
64. $y = \frac{1}{6}(1 + \cos^2(7t))^3$

65. $y = \sqrt{1 + \cos(t^2)}$ 66. $y = 4 \sin(\sqrt{1 + \sqrt{t}})$
 67. $y = \tan^2(\sin^3 t)$ 68. $y = \cos^4(\sec^2 3t)$
 69. $y = 3t(2t^2 - 5)^4$ 70. $y = \sqrt{3t + \sqrt{2 + \sqrt{1 - t}}}$

Second DerivativesFind y'' in Exercises 71–78.

71. $y = \left(1 + \frac{1}{x}\right)^3$ 72. $y = (1 - \sqrt{x})^{-1}$
 73. $y = \frac{1}{9} \cot(3x - 1)$ 74. $y = 9 \tan\left(\frac{x}{3}\right)$
 75. $y = x(2x + 1)^4$ 76. $y = x^2(x^3 - 1)^5$
 77. $y = e^{x^2} + 5x$ 78. $y = \sin(x^2 e^x)$

For each of the following functions, solve both $f'(x) = 0$ and $f''(x) = 0$ for x .

79. $f(x) = x(x - 4)^3$
 80. $f(x) = \sec^2 x - 2 \tan x$ for $0 \leq x \leq 2\pi$

Finding Derivative ValuesIn Exercises 81–86, find the value of $(f \circ g)'$ at the given value of x .

81. $f(u) = u^5 + 1$, $u = g(x) = \sqrt{x}$, $x = 1$
 82. $f(u) = 1 - \frac{1}{u}$, $u = g(x) = \frac{1}{1 - x}$, $x = -1$
 83. $f(u) = \cot \frac{\pi u}{10}$, $u = g(x) = 5\sqrt{x}$, $x = 1$
 84. $f(u) = u + \frac{1}{\cos^2 u}$, $u = g(x) = \pi x$, $x = 1/4$
 85. $f(u) = \frac{2u}{u^2 + 1}$, $u = g(x) = 10x^2 + x + 1$, $x = 0$
 86. $f(u) = \left(\frac{u - 1}{u + 1}\right)^2$, $u = g(x) = \frac{1}{x^2} - 1$, $x = -1$
 87. Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$, and $y = f(g(x))$. What is y' at $x = 2$?
 88. If $r = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is dr/dt at $t = 0$?
 89. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	$1/3$	-3
3	3	-4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x .

- a. $2f(x)$, $x = 2$ b. $f(x) + g(x)$, $x = 3$
 c. $f(x) \cdot g(x)$, $x = 3$ d. $f(x)/g(x)$, $x = 2$
 e. $f(g(x))$, $x = 2$ f. $\sqrt{f(x)}$, $x = 2$
 g. $1/g^2(x)$, $x = 3$ h. $\sqrt{f^2(x) + g^2(x)}$, $x = 2$
 90. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	$1/3$
1	3	-4	$-1/3$	$-8/3$

Find the derivatives with respect to x of the following combinations at the given value of x .

- a. $5f(x) - g(x)$, $x = 1$ b. $f(x)g^3(x)$, $x = 0$
 c. $\frac{f(x)}{g(x) + 1}$, $x = 1$ d. $f(g(x))$, $x = 0$
 e. $g(f(x))$, $x = 0$ f. $(x^{11} + f(x))^{-2}$, $x = 1$
 g. $f(x + g(x))$, $x = 0$
 91. Find ds/dt when $\theta = 3\pi/2$ if $s = \cos \theta$ and $d\theta/dt = 5$.
 92. Find dy/dt when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

Theory and Examples

What happens if you can write a function as a composition in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 93 and 94.

93. Find dy/dx if $y = x$ by using the Chain Rule with y as a composition of
 a. $y = (u/5) + 7$ and $u = 5x - 35$
 b. $y = 1 + (1/u)$ and $u = 1/(x - 1)$.
 94. Find dy/dx if $y = x^{3/2}$ by using the Chain Rule with y as a composition of
 a. $y = u^3$ and $u = \sqrt{x}$
 b. $y = \sqrt{u}$ and $u = x^3$.
 95. Find the tangent line to $y = ((x - 1)/(x + 1))^2$ at $x = 0$.
 96. Find the tangent line to $y = \sqrt{x^2 - x + 7}$ at $x = 2$.
 97. a. Find the tangent line to the curve $y = 2 \tan(\pi x/4)$ at $x = 1$.
 b. **Slopes on a tangent curve** What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.
 98. **Slopes on sine curves**
 a. Find equations for the tangent lines to the curves $y = \sin 2x$ and $y = -\sin(x/2)$ at the origin. Is there anything special about how the tangent lines are related? Give reasons for your answer.
 b. Can anything be said about the tangent lines to the curves $y = \sin mx$ and $y = -\sin(x/m)$ at the origin (m a constant $\neq 0$)? Give reasons for your answer.
 c. For a given m , what are the largest values the slopes of the curves $y = \sin mx$ and $y = -\sin(x/m)$ can ever have? Give reasons for your answer.
 d. The function $y = \sin x$ completes one period on the interval $[0, 2\pi]$, the function $y = \sin 2x$ completes two periods, the function $y = \sin(x/2)$ completes half a period, and so on. Is there any relation between the number of periods $y = \sin mx$ completes on $[0, 2\pi]$ and the slope of the curve $y = \sin mx$ at the origin? Give reasons for your answer.

99. **Running machinery too fast** Suppose that a piston is moving straight up and down and that its position at time t seconds is

$$s = A \cos(2\pi bt),$$

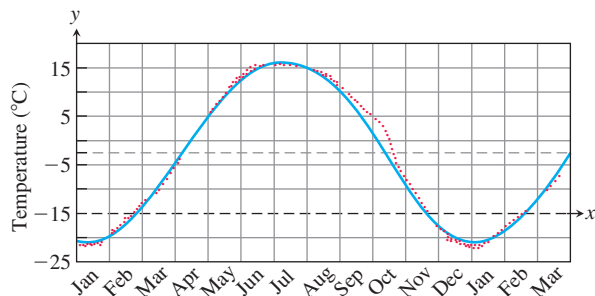
with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why some machinery breaks when you run it too fast.)

- 100. Temperatures in Fairbanks, Alaska** The graph in the accompanying figure shows the average Celsius temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 20 \sin \left[\frac{2\pi}{365} (x - 101) \right] - 4$$

and is graphed in the accompanying figure.

- On what day is the temperature increasing the fastest?
- About how many degrees per day is the temperature increasing when it is increasing at its fastest?



- 101. Particle motion** The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ s.
- 102. Constant acceleration** Suppose that the velocity of a falling body is $v = k\sqrt{s}$ m/s (k a constant) at the instant the body has fallen s meters from its starting point. Show that the body's acceleration is constant.
- 103. Falling meteorite** The velocity of a heavy meteorite entering Earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s km from Earth's center. Show that the meteorite's acceleration is inversely proportional to s^2 .
- 104. Particle acceleration** A particle moves along the x -axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.
- 105. Temperature and the period of a pendulum** For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi \sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L . In symbols, with u being temperature and k the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

- 106. Chain Rule** Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the compositions

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

- T 107. The derivative of $\sin 2x$** Graph the function $y = 2 \cos 2x$ for $-2 \leq x \leq 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x + h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and 0.2 . Experiment with other values of h , including negative values. What do you see happening as $h \rightarrow 0$? Explain this behavior.

- T 108. The derivative of $\cos(x^2)$** Graph $y = -2x \sin(x^2)$ for $-2 \leq x \leq 3$. Then, on the same screen, graph

$$y = \frac{\cos((x + h)^2) - \cos(x^2)}{h}$$

for $h = 1.0, 0.7$, and 0.3 . Experiment with other values of h . What do you see happening as $h \rightarrow 0$? Explain this behavior.

Using the Chain Rule, show that the Power Rule $(d/dx)x^n = nx^{n-1}$ holds for the functions x^n in Exercises 109 and 110.

109. $x^{1/4} = \sqrt[4]{x}$

110. $x^{3/4} = \sqrt[4]{x^3}$

- 111.** Consider the function

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x > 0 \\ 0, & x \leq 0 \end{cases}$$

- Show that f is continuous at $x = 0$.
- Determine f' for $x \neq 0$.
- Show that f is not differentiable at $x = 0$.

- 112.** Consider the function

$$f(x) = \begin{cases} x^2 \cos\left(\frac{2}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

- Show that f is continuous at $x = 0$.
- Determine f' for $x \neq 0$.
- Show that f is not differentiable at $x = 0$.
- Show that f' is not continuous at $x = 0$.

- 113.** Verify each of the following statements.

- If f is even, then f' is odd.
- If f is odd, then f' is even.

COMPUTER EXPLORATIONS

Trigonometric Polynomials

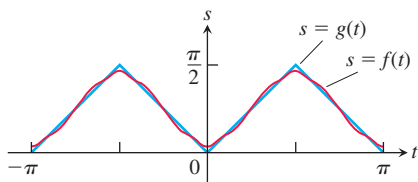
- 114.** As the accompanying figure shows, the trigonometric "polynomial"

$$s = f(t) = 0.78540 - 0.63662 \cos 2t - 0.07074 \cos 6t - 0.02546 \cos 10t - 0.01299 \cos 14t$$

gives a good approximation of the sawtooth function $s = g(t)$ on the interval $[-\pi, \pi]$. How well does the derivative of f approximate the derivative of g at the points where dg/dt is defined? To find out, carry out the following steps.

- Graph dg/dt (where defined) over $[-\pi, \pi]$.
- Find df/dt .

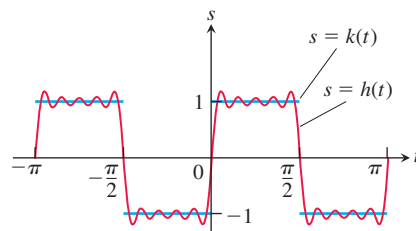
- c. Graph df/dt . Where does the approximation of dg/dt by df/dt seem to be best? Least good? Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them, as we see in the next exercise.



115. (Continuation of Exercise 114.) In Exercise 114, the trigonometric polynomial $f(t)$ that approximated the sawtooth function $g(t)$ on $[-\pi, \pi]$ had a derivative that approximated the derivative of the sawtooth function. It is possible, however, for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the trigonometric “polynomial”

$$s = h(t) = 1.2732 \sin 2t + 0.4244 \sin 6t + 0.25465 \sin 10t \\ + 0.18189 \sin 14t + 0.14147 \sin 18t$$

graphed in the accompanying figure approximates the step function $s = k(t)$ shown there. Yet the derivative of h is nothing like the derivative of k .



- Graph dk/dt (where defined) over $[-\pi, \pi]$.
- Find dh/dt .
- Graph dh/dt to see how badly the graph fits the graph of dk/dt . Comment on what you see.

3.7 Implicit Differentiation

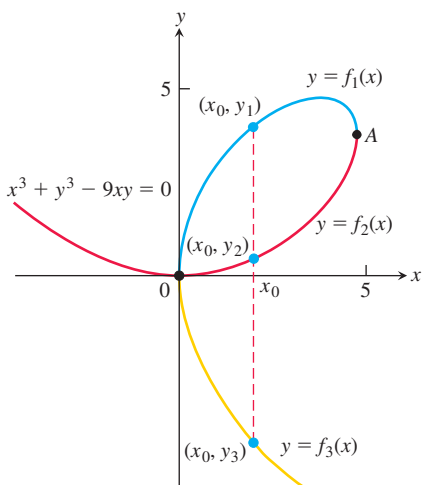


FIGURE 3.29 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x . The curve can, however, be divided into separate arcs that are the graphs of functions of x . This particular curve, called a *folium*, dates to Descartes in 1638.

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x . We have learned rules for differentiating functions defined in this way. A different situation occurs when we encounter equations like

$$x^3 + y^3 - 9xy = 0, \quad y^2 - x = 0, \quad \text{or} \quad x^2 + y^2 - 25 = 0.$$

(See Figures 3.29, 3.30, and 3.31.) Each of these equations defines an *implicit* relation between the variables x and y , meaning that a value of x may determine more than one value of y , even though we do not have a simple formula for the y -values. In some cases we may be able to solve such an equation for y as an explicit function (or even several functions) of x . When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by *implicit differentiation*. This section describes the technique.

Implicitly Defined Functions

We begin with examples involving familiar equations that we can solve for y as a function of x and then calculate dy/dx in the usual way. Then we differentiate the equations implicitly, and find the derivative. We will see that the two methods give the same answer. Following the examples, we summarize the steps involved in the new method. In the examples and exercises, it is always assumed that the given equation determines y implicitly as a differentiable function of x so that dy/dx exists.

EXAMPLE 1 Find dy/dx if $y^2 = x$.

Solution The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (Figure 3.30). We know how to calculate the derivative of each of these for $x > 0$:

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

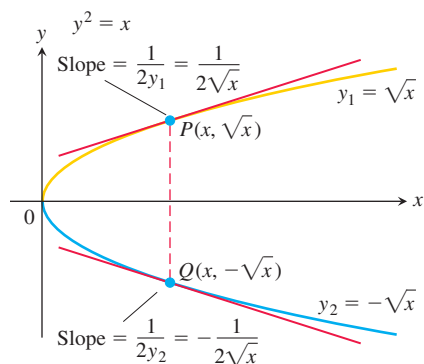


FIGURE 3.30 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x > 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y .

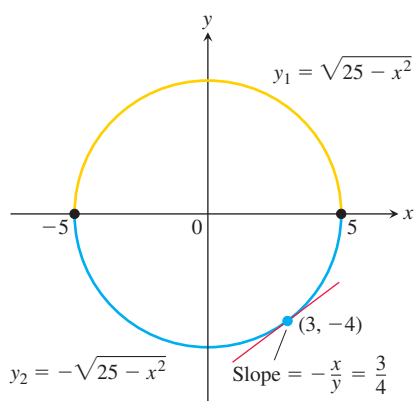


FIGURE 3.31 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$.

But suppose that we knew only that the equation $y^2 = x$ defined y as one or more differentiable functions of x for $x > 0$ without knowing exactly what these functions were. Can we still find dy/dx ?

The answer is yes. To find dy/dx , we simply differentiate both sides of the equation $y^2 = x$ with respect to x , treating $y = f(x)$ as a differentiable function of x :

$$\begin{aligned} y^2 &= x \\ 2y \frac{dy}{dx} &= 1 && \text{The Chain Rule gives } \frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y \frac{dy}{dx}. \\ \frac{dy}{dx} &= \frac{1}{2y}. \end{aligned}$$

This one formula gives the derivatives we calculated for *both* explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}.$$

EXAMPLE 2 Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution The circle is not the graph of a single function of x . Rather, it is the combined graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 3.31). The point $(3, -4)$ lies on the graph of y_2 , so we can find the slope by calculating the derivative directly, using the Power Chain Rule:

$$\left. \frac{dy_2}{dx} \right|_{x=3} = \left. -\frac{2x}{2\sqrt{25 - x^2}} \right|_{x=3} = \left. -\frac{6}{2\sqrt{25 - 9}} \right|_{x=3} = \frac{3}{4}. \quad \frac{d}{dx}(-(25 - x^2)^{1/2}) = -\frac{1}{2}(25 - x^2)^{-1/2}(-2x)$$

We can solve this problem more easily by differentiating the given equation of the circle implicitly with respect to x :

$$\begin{aligned} \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(25) \\ 2x + 2y \frac{dy}{dx} &= 0 && \text{See Example 1.} \\ \frac{dy}{dx} &= -\frac{x}{y}. \end{aligned}$$

$$\text{The slope at } (3, -4) \text{ is } \left. -\frac{x}{y} \right|_{(3, -4)} = -\frac{3}{-4} = \frac{3}{4}.$$

Notice that unlike the slope formula for dy_2/dx , which applies only to points below the x -axis, the formula $dy/dx = -x/y$ applies everywhere the circle has a slope—that is, at all circle points (x, y) where $y \neq 0$. Notice also that the derivative involves *both* variables x and y , not just the independent variable x .

To calculate the derivatives of other implicitly defined functions, we proceed as in Examples 1 and 2: We treat y as a differentiable implicit function of x and apply the usual rules to differentiate both sides of the defining equation.

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x , treating y as a differentiable function of x .
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx .

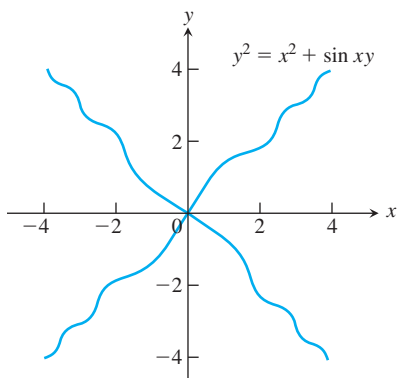


FIGURE 3.32 The graph of the equation in Example 3.

EXAMPLE 3 Find dy/dx if $y^2 = x^2 + \sin xy$ (Figure 3.32).

Solution We differentiate the equation implicitly.

$$\begin{aligned}
 y^2 &= x^2 + \sin xy \\
 \frac{d}{dx}(y^2) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy) && \text{Differentiate both sides with respect to } x \dots \\
 2y \frac{dy}{dx} &= 2x + (\cos xy) \frac{d}{dx}(xy) && \dots \text{treating } y \text{ as a function of } x \text{ and using the Chain Rule.} \\
 2y \frac{dy}{dx} &= 2x + (\cos xy) \left(y + x \frac{dy}{dx} \right) && \text{Treat } xy \text{ as a product.} \\
 2y \frac{dy}{dx} - (\cos xy) \left(x \frac{dy}{dx} \right) &= 2x + (\cos xy) y && \text{Collect terms with } dy/dx. \\
 (2y - x \cos xy) \frac{dy}{dx} &= 2x + y \cos xy \\
 \frac{dy}{dx} &= \frac{2x + y \cos xy}{2y - x \cos xy} && \text{Solve for } dy/dx.
 \end{aligned}$$

Notice that the formula for dy/dx applies everywhere that the implicitly defined curve has a slope. Notice again that the derivative involves *both* variables x and y , not just the independent variable x . ■

Derivatives of Higher Order

Implicit differentiation can also be used to find higher derivatives.

EXAMPLE 4 Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\begin{aligned}
 \frac{d}{dx}(2x^3 - 3y^2) &= \frac{d}{dx}(8) && \text{Treat } y \text{ as a function of } x. \\
 6x^2 - 6yy' &= 0 \\
 y' &= \frac{x^2}{y}, \quad \text{when } y \neq 0 && \text{Solve for } y'.
 \end{aligned}$$

We now apply the Quotient Rule to find y'' .

$$y'' = \frac{d}{dx} \left(\frac{x^2}{y} \right) = \frac{2xy - x^2 y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y .

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2} \left(\frac{x^2}{y} \right) = \frac{2x}{y} - \frac{x^4}{y^3}, \quad \text{when } y \neq 0$$

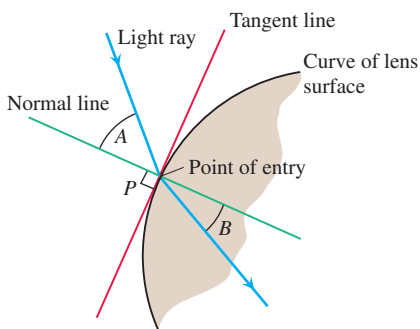


FIGURE 3.33 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

Lenses, Tangent Lines, and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in Figure 3.33). This line is called the *normal line* to the surface at the point of entry. In a profile view of a lens like the one in Figure 3.33, the **normal line** is the line perpendicular (also said to be *orthogonal*) to the tangent line of the profile curve at the point of entry.

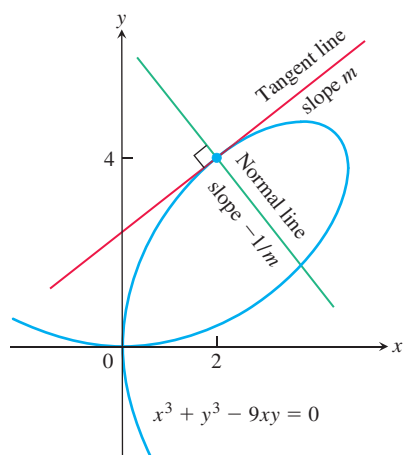


FIGURE 3.34 Example 5 shows how to find equations for the tangent line and normal line to the folium of Descartes at $(2, 4)$.

EXAMPLE 5 Show that the point $(2, 4)$ lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent line and normal line to the curve there (Figure 3.34).

Solution The point $(2, 4)$ lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9(2)(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at $(2, 4)$, we first use implicit differentiation to find a formula for dy/dx :

$$\begin{aligned}
 x^3 + y^3 - 9xy &= 0 \\
 \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) &= \frac{d}{dx}(0) && \text{Differentiate both sides} \\
 &&& \text{with respect to } x. \\
 3x^2 + 3y^2 \frac{dy}{dx} - 9\left(x \frac{dy}{dx} + \frac{dx}{dx}y\right) &= 0 && \text{Treat } xy \text{ as a product} \\
 &&& \text{and } y \text{ as a function of } x. \\
 (3y^2 - 9x) \frac{dy}{dx} + 3x^2 - 9y &= 0 \\
 3(y^2 - 3x) \frac{dy}{dx} &= 9y - 3x^2 \\
 \frac{dy}{dx} &= \frac{3y - x^2}{y^2 - 3x}. && \text{Solve for } dy/dx.
 \end{aligned}$$

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left. \frac{dy}{dx} \right|_{(2,4)} = \left. \frac{3y - x^2}{y^2 - 3x} \right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent line at $(2, 4)$ is the line through $(2, 4)$ with slope $4/5$:

$$\begin{aligned}
 y &= 4 + \frac{4}{5}(x - 2) \\
 y &= \frac{4}{5}x + \frac{12}{5}.
 \end{aligned}$$

The normal line to the curve at $(2, 4)$ is the line perpendicular to the tangent line there, the line through $(2, 4)$ with slope $-5/4$:

$$\begin{aligned}
 y &= 4 - \frac{5}{4}(x - 2) \\
 y &= -\frac{5}{4}x + \frac{13}{2}.
 \end{aligned}$$

Slopes of two nonvertical perpendicular lines are negative reciprocals of each other (see Appendix A.4). ■

EXERCISES 3.7

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 1–16.

1. $x^2y + xy^2 = 6$
2. $x^3 + y^3 = 18xy$
3. $2xy + y^2 = x + y$
4. $x^3 - xy + y^3 = 1$
5. $x^2(x - y)^2 = x^2 - y^2$
6. $(3xy + 7)^2 = 6y$
7. $y^2 = \frac{x-1}{x+1}$
8. $x^3 = \frac{2x-y}{x+3y}$
9. $x = \sec y$
10. $xy = \cot(xy)$
11. $x + \tan(xy) = 0$
12. $x^4 + \sin y = x^3y^2$
13. $y \sin\left(\frac{1}{y}\right) = 1 - xy$
14. $x \cos(2x + 3y) = y \sin x$
15. $e^{2x} = \sin(x + 3y)$
16. $e^{x^2y} = 2x + 2y$

Find $dr/d\theta$ in Exercises 17–20.

17. $\theta^{1/2} + r^{1/2} = 1$
18. $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$
19. $\sin(r\theta) = \frac{1}{2}$
20. $\cos r + \cot \theta = e^{r\theta}$

Second Derivatives

In Exercises 21–28, use implicit differentiation to find dy/dx and then d^2y/dx^2 . Write the solutions in terms of x and y only.

21. $x^2 + y^2 = 1$
22. $x^{2/3} + y^{2/3} = 1$
23. $y^2 = e^{x^2} + 2x$
24. $y^2 - 2x = 1 - 2y$
25. $2\sqrt{y} = x - y$
26. $xy + y^2 = 1$
27. $3 + \sin y = y - x^3$
28. $\sin y = x \cos y - 2$

29. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.
 30. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

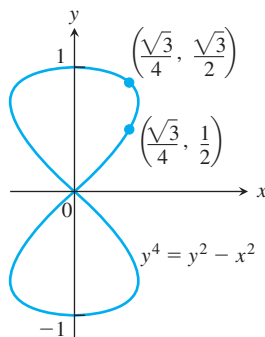
In Exercises 31 and 32, find the slope of the curve at the given points.

31. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$
 32. $(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1)$

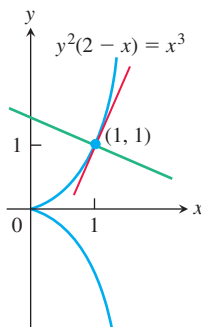
Slopes, Tangent Lines, and Normal Lines

In Exercises 33–42, verify that the given point is on the curve and find the lines that are (a) tangent and (b) normal to the curve at the given point.

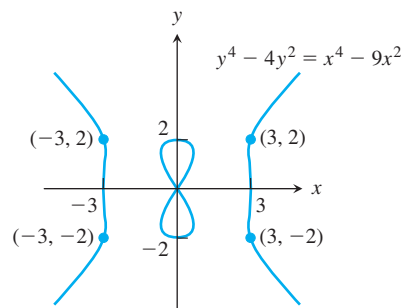
33. $x^2 + xy - y^2 = 1$, $(2, 3)$
 34. $x^2 + y^2 = 25$, $(3, -4)$
 35. $x^2y^2 = 9$, $(-1, 3)$
 36. $y^2 - 2x - 4y - 1 = 0$, $(-2, 1)$
 37. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$, $(-1, 0)$
 38. $x^2 - \sqrt{3}xy + 2y^2 = 5$, $(\sqrt{3}, 2)$
 39. $2xy + \pi \sin y = 2\pi$, $(1, \pi/2)$
 40. $x \sin 2y = y \cos 2x$, $(\pi/4, \pi/2)$
 41. $y = 2 \sin(\pi x - y)$, $(1, 0)$
 42. $x^2 \cos^2 y - \sin y = 0$, $(0, \pi)$
 43. **Parallel tangent lines** Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x -axis, and show that the tangent lines to the curve at these points are parallel. What is the common slope of these tangent lines?
 44. **Normal lines parallel to a line** Find the normal lines to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.
 45. **The eight curve** Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



46. **The cissoid of Diocles (from about 200 B.C.)** Find equations for the tangent line and normal line to the cissoid of Diocles $y^2(2 - x) = x^3$ at $(1, 1)$.



47. **The devil's curve (Gabriel Cramer, 1750)** Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



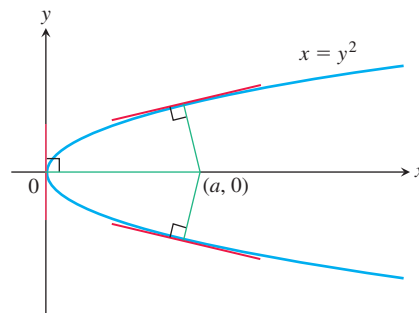
48. **The folium of Descartes** (See Figure 3.29)
 a. Find the slope of the folium of Descartes $x^3 + y^3 - 9xy = 0$ at the points $(4, 2)$ and $(2, 4)$.
 b. At what point other than the origin does the folium have a horizontal tangent line?
 c. Find the coordinates of the point A in Figure 3.29 where the folium has a vertical tangent line.

Theory and Examples

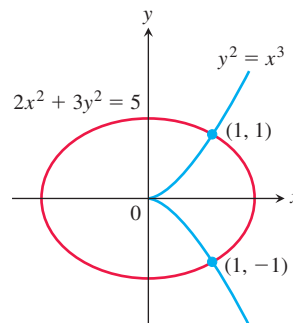
49. **Intersecting normal line** The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$ intersects the curve at what other point?
 50. **Power rule for rational exponents** Let p and q be integers with $q > 0$. If $y = x^{p/q}$, differentiate the equivalent equation $y^q = x^p$ implicitly and show that, for $y \neq 0$,

$$\frac{d}{dx} x^{p/q} = \frac{p}{q} x^{(p/q)-1}.$$

51. **Normal lines to a parabola** Show that if it is possible to draw three normal lines from the point $(a, 0)$ to the parabola $x = y^2$ shown in the accompanying diagram, then a must be greater than $1/2$. One of the normal lines is the x -axis. For what value of a are the other two normal lines perpendicular?



52. Is there anything special about the tangent lines to the curves $y^2 = x^3$ and $2x^2 + 3y^2 = 5$ at the points $(1, \pm 1)$? Give reasons for your answer.

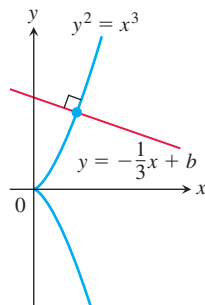


53. Verify that the following pairs of curves meet orthogonally.

a. $x^2 + y^2 = 4$, $x^2 = 3y^2$

b. $x = 1 - y^2$, $x = \frac{1}{3}y^2$

54. The graph of $y^2 = x^3$ is called a **semicubical parabola** and is shown in the accompanying figure. Determine the constant b so that the line $y = -\frac{1}{3}x + b$ meets this graph orthogonally.



In Exercises 55 and 56, find both dy/dx (treating y as a differentiable function of x) and dx/dy (treating x as a differentiable function of y). How do dy/dx and dx/dy seem to be related?

55. $xy^3 + x^2y = 6$

56. $x^3 + y^2 = \sin^2 y$

57. **Derivative of arcsine** Assume that $y = \sin^{-1}x$ is a differentiable function of x . By differentiating the equation $x = \sin y$ implicitly, show that $dy/dx = 1/\sqrt{1-x^2}$.

58. Use the formula in Exercise 57 to find dy/dx if

a. $y = (\sin^{-1}x)^2$

b. $y = \sin^{-1}\left(\frac{1}{x}\right)$.

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps in Exercises 59–66.

a. Plot the equation with the implicit plotter of a CAS. Check to see that the given point P satisfies the equation.

b. Using implicit differentiation, find a formula for the derivative dy/dx and evaluate it at the given point P .

c. Use the slope found in part (b) to find an equation for the tangent line to the curve at P . Then plot the implicit curve and tangent line together on a single graph.

59. $x^3 - xy + y^3 = 7$, $P(2, 1)$

60. $x^5 + y^3x + yx^2 + y^4 = 4$, $P(1, 1)$

61. $y^2 + y = \frac{2+x}{1-x}$, $P(0, 1)$

62. $y^3 + \cos xy = x^2$, $P(1, 0)$

63. $x + \tan\left(\frac{y}{x}\right) = 2$, $P\left(1, \frac{\pi}{4}\right)$

64. $xy^3 + \tan(x+y) = 1$, $P\left(\frac{\pi}{4}, 0\right)$

65. $2y^2 + (xy)^{1/3} = x^2 + 2$, $P(1, 1)$

66. $x\sqrt{1+2y} + y = x^2$, $P(1, 0)$

3.8 Derivatives of Inverse Functions and Logarithms

In Section 1.5 we saw how the inverse of a function undoes, or inverts, the effect of that function. We defined there the natural logarithm function $f^{-1}(x) = \ln x$ as the inverse of the natural exponential function $f(x) = e^x$. This is one of the most important function-inverse pairs in mathematics and science. We learned how to differentiate the exponential function in Section 3.3. Here we develop a rule for differentiating the inverse of a differentiable function, and we apply the rule to find the derivative of the natural logarithm function.

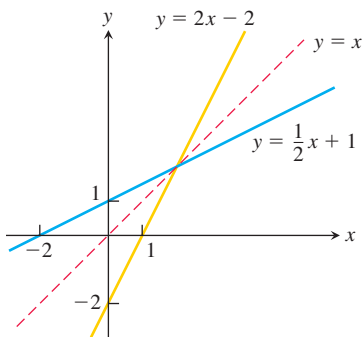


FIGURE 3.35 Graphing a line and its inverse together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other.

Derivatives of Inverses of Differentiable Functions

We calculated the inverse of the function $f(x) = (1/2)x + 1$ to be $f^{-1}(x) = 2x - 2$ in Example 3 of Section 1.5. Figure 3.35 shows the graphs of both functions. If we calculate their derivatives, we see that

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2} \\ \frac{d}{dx}f^{-1}(x) &= \frac{d}{dx}(2x - 2) = 2.\end{aligned}$$

The derivatives are reciprocals of one another, so the slope of one line is the reciprocal of the slope of its inverse line. (See Figure 3.35.)

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$, the reflected line has slope $1/m$.

64. Use the identity

$$\operatorname{arccot} x = \frac{\pi}{2} - \arctan x$$

to derive the formula for the derivative of $\operatorname{arccot} x$ in Table 3.1 from the formula for the derivative of $\arctan x$.

65. What is special about the functions

$$f(x) = \sin^{-1} \frac{x-1}{x+1}, x \geq 0, \text{ and } g(x) = 2 \tan^{-1} \sqrt{x}?$$

Explain.

66. What is special about the functions

$$f(x) = \sin^{-1} \frac{1}{\sqrt{x^2+1}} \text{ and } g(x) = \tan^{-1} \frac{1}{x}?$$

Explain.

T 67. Find the values of

a. $\operatorname{arcsec} 1.5$ b. $\csc^{-1}(-1.5)$ c. $\operatorname{arccot} 2$

T 68. Find the values of

a. $\sec^{-1}(-3)$ b. $\operatorname{arccsc} 1.7$ c. $\operatorname{arccot}(-2)$

T In Exercises 69–71, find the domain and range of each composite function. Then graph the composition of the two functions on separate

screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

69. a. $y = \arctan(\tan x)$ b. $y = \tan(\arctan x)$

70. a. $y = \arcsin(\sin x)$ b. $y = \sin(\arcsin x)$

71. a. $y = \arccos(\cos x)$ b. $y = \cos(\arccos x)$

T Use your graphing utility for Exercises 72–76.

72. Graph $y = \sec(\sec^{-1} x) = \sec(\cos^{-1}(1/x))$. Explain what you see.

73. **Newton's serpentine** Graph Newton's serpentine, $y = 4x/(x^2 + 1)$. Then graph $y = 2 \sin(2 \tan^{-1} x)$ in the same graphing window. What do you see? Explain.

74. Graph the rational function $y = (2 - x^2)/x^2$. Then graph $y = \cos(2 \sec^{-1} x)$ in the same graphing window. What do you see? Explain.

75. Graph $f(x) = \arcsin x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f'' .

76. Graph $f(x) = \arctan x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f'' .

3.10 Related Rates

In this section we look at questions that arise when two or more related quantities are changing. The problem of determining how the rate of change of one of them affects the rates of change of the others is called a *related rates problem*.

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3} \pi r^3.$$

Using the Chain Rule, we differentiate both sides with respect to t to find an equation relating the rates of change of V and r ,

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant. Note that it is easier to directly measure the rate of increase of the volume (the rate at which air is being pumped into the balloon) than it is to measure the increase in the radius. The related rates equation allows us to calculate dr/dt from dV/dt .

Very often the key to relating the variables in a related rates problem is drawing a picture that shows the geometric relations between them, as illustrated in the following example.

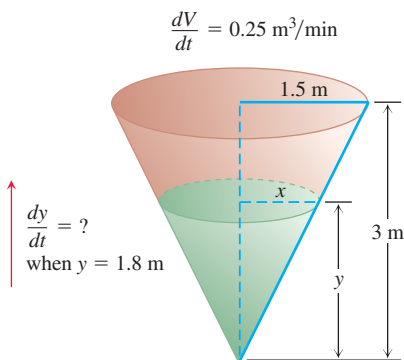


FIGURE 3.45 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

EXAMPLE 1 Water runs into a conical tank at the rate of $0.25 \text{ m}^3/\text{min}$. The tank stands point down and has a height of 3 m and a base radius of 1.5 m. How fast is the water level rising when the water is 1.8 m deep?

Solution Figure 3.45 shows a partially filled conical tank. The variables in the problem are

V = volume (m^3) of the water in the tank at time t (min)

x = radius (m) of the surface of the water at time t

y = depth (m) of the water in the tank at time t .

We assume that V , x , and y are differentiable functions of t . The constants are the dimensions of the tank. We are asked for dy/dt when

$$y = 1.8 \text{ m} \quad \text{and} \quad \frac{dV}{dt} = 0.25 \text{ m}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves x as well as V and y . Because no information is given about x and dx/dt at the time in question, we need to eliminate x . The similar triangles in Figure 3.45 give us a way to express x in terms y :

$$\frac{x}{y} = \frac{1.5}{3} \quad \text{or} \quad x = \frac{y}{2}.$$

Therefore, we find

$$V = \frac{1}{3}\pi \left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3y^2 \frac{dy}{dt} = \frac{\pi}{4}y^2 \frac{dy}{dt}.$$

Finally, use $y = 1.8$ and $dV/dt = 0.25$ to solve for dy/dt .

$$\begin{aligned} 0.25 &= \frac{\pi}{4}(1.8)^2 \frac{dy}{dt} \\ \frac{dy}{dt} &= \frac{1}{3.24\pi} \approx 0.098 \end{aligned}$$

At the moment in question, the water level is rising at about 0.098 m/min. ■

Related Rates Problem Strategy

1. Let t denote time, and **choose names** for all of the variables that change over time (we will assume that those variables are differentiable functions of t). Identify any quantities that remain constant (these do not need to be given names). In most problems it will be very helpful to **draw a picture** that depicts the setup of the problem.
2. Write an **equation that relates the variables** (and any constants that are present). You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates or values you know.
3. Differentiate with respect to t to obtain a **related rates equation**.
4. Substitute all of the numerical values provided in the problem into the related rates equation. You may need to use the equation(s) relating the variables (which you obtained in Step 2), or use other relationships (such as trigonometric identities), until you reach the point at which *the only remaining unknown quantity is the rate of change that you are asked to find*. Solve for this unknown.

EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

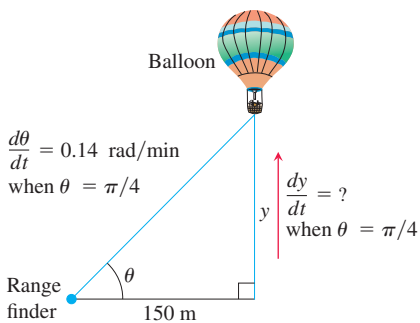


FIGURE 3.46 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

Solution We draw a picture (Figure 3.46) and name the variables that appear in the problem (which we assume are differentiable functions of t , where time is measured in minutes):

- θ = the angle in radians the range finder makes with the ground, and
- y = the height in meters of the balloon above the ground.

One constant in the picture is the distance from the range finder to the liftoff point (150 m). There is no need to give this distance a special symbol.

Trigonometry (Section 1.3) yields

$$\frac{y}{150} = \tan \theta \quad \text{or} \quad y = 150 \tan \theta. \quad \text{Equation relating the variables}$$

By differentiating with respect to t using the Chain Rule, we obtain

$$\frac{dy}{dt} = 150(\sec^2 \theta) \frac{d\theta}{dt}. \quad \text{Related rates equation}$$

Substituting the known values $\frac{d\theta}{dt} = 0.14$ rad/min and $\theta = \frac{\pi}{4}$ gives

$$\frac{dy}{dt} = 150 \left(\sec^2 \frac{\pi}{4} \right) (0.14) = 150 (\sqrt{2})^2 (0.14) = 42. \quad \sec \frac{\pi}{4} = \sqrt{2}$$

Thus, at the moment in question, the balloon is rising at the rate of 42 m/min. ■

EXAMPLE 3 A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 km north of the intersection and the car is 0.8 km to the east, the police determine with radar that the distance between them and the car is increasing at 30 km/h. If the cruiser is moving at 100 km/h at the instant of measurement, what is the speed of the car?

Solution We picture the car and cruiser in the coordinate plane, using the positive x -axis as the eastbound highway and the positive y -axis as the southbound highway (Figure 3.47). We let t represent time and set

- x = position of car at time t
- y = position of cruiser at time t
- s = distance between car and cruiser at time t

We assume that x , y , and s are differentiable functions of t .

We want to find dx/dt when

$$x = 0.8 \text{ km}, \quad y = 0.6 \text{ km}, \quad \frac{dy}{dt} = -100 \text{ km/h}, \quad \frac{ds}{dt} = 30 \text{ km/h}.$$

Note that dy/dt is negative because y is decreasing.

We differentiate the distance equation between the car and the cruiser,

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$\begin{aligned} 2s \frac{ds}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \\ \frac{ds}{dt} &= \frac{1}{s} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) = \frac{1}{\sqrt{x^2 + y^2}} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right). \end{aligned}$$

Finally, we use $x = 0.8$, $y = 0.6$, $dy/dt = -100$, $ds/dt = 30$, and solve for dx/dt .

$$\begin{aligned} 30 &= \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}} \left(0.8 \frac{dx}{dt} + (0.6)(-100) \right) \\ \frac{dx}{dt} &= \frac{30\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(100)}{0.8} = 112.5 \end{aligned}$$

At the moment in question, the car's speed is 112.5 km/h. ■

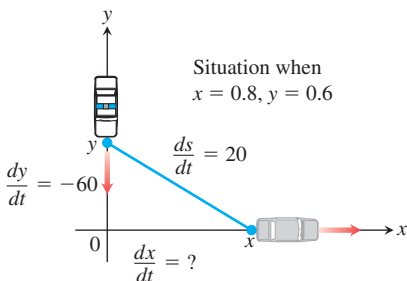


FIGURE 3.47 The speed of the car is related to the speed of the police cruiser and the rate of change of the distance s between them (Example 3).

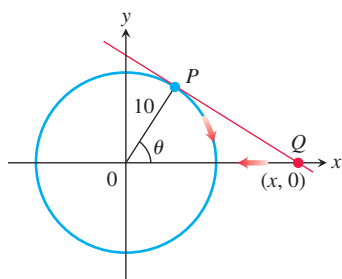


FIGURE 3.48 The particle P travels clockwise along the circle (Example 4).

EXAMPLE 4 A particle P moves clockwise at a constant rate along a circle of radius 10 m centered at the origin. The particle's initial position is $(0, 10)$ on the y -axis, and its final destination is the point $(10, 0)$ on the x -axis. Once the particle is in motion, the tangent line at P intersects the x -axis at a point Q (which moves over time). If it takes the particle 30 s to travel from start to finish, how fast is the point Q moving along the x -axis when it is 20 m from the center of the circle?

Solution We picture the situation in the coordinate plane with the circle centered at the origin (see Figure 3.48). We let t represent time and let θ denote the angle in radians from the x -axis to the radial line joining the origin to P . Since the particle travels from start to finish in 30 s, it is traveling along the circle at a constant rate of $\pi/2$ radians in $1/2$ min, or π rad/min. In other words, $d\theta/dt = -\pi$, with t being measured in minutes. The negative sign appears because θ is decreasing over time.

Setting $x(t)$ to be the distance in meters at time t from the point Q to the origin, we see from Figure 3.48 that

$$x \cos \theta = 10. \quad \text{Equation relating the variables}$$

Differentiation with respect to t gives

$$x(-\sin \theta) \frac{d\theta}{dt} + \frac{dx}{dt} \cos \theta = 0. \quad \text{Related rates equation}$$

We want to find dx/dt given that $x = 20$ and $d\theta/dt = -\pi$. The equation $x \cos \theta = 10$ implies $\cos \theta = 10/20 = 1/2$. Furthermore, for angles θ in the first quadrant, the identity $\sin^2 \theta + \cos^2 \theta = 1$ yields $\sin \theta = \sqrt{1 - (1/2)^2} = \sqrt{3}/2$. Substituting into the related rates equation, we obtain

$$\begin{aligned} (20) \left(-\frac{\sqrt{3}}{2} \right) (-\pi) + \frac{dx}{dt} \cdot \frac{1}{2} &= 0 \\ \frac{dx}{dt} &= -20\sqrt{3}\pi. \end{aligned}$$

Note that x is decreasing because dx/dt is negative. At the moment in question, the point Q is moving toward the origin at the speed of $20\sqrt{3}\pi \approx 109$ m/min. ■

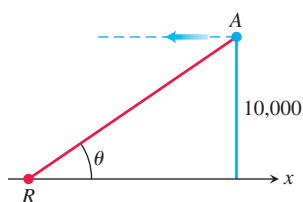


FIGURE 3.49 Jet airliner A traveling at constant altitude toward radar station R (Example 5).

EXAMPLE 5 A jet airliner is flying at a constant altitude of 10,000 m above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30° . How fast (in kilometers per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise) at the rate of $1/3$ deg/s in order to keep the aircraft within its direct line of sight?

Solution The aircraft A and radar station R are pictured in the coordinate plane, using the positive x -axis as the horizontal distance at sea level from R to A , and the positive y -axis as the vertical altitude above sea level. We let t represent time and observe that $y = 10,000$ is a constant. The general situation and line-of-sight angle θ are depicted in Figure 3.49. We want to find dx/dt when $\theta = \pi/6$ rad and $d\theta/dt = 1/3$ deg/s.

From Figure 3.49, we see that

$$\frac{10,000}{x} = \tan \theta \quad \text{or} \quad x = 10,000 \cot \theta.$$

Using kilometers instead of meters for our distance units, the last equation translates to

$$x = \frac{10,000}{1000} \cot \theta.$$

Differentiation with respect to t gives

$$\frac{dx}{dt} = -10 \csc^2 \theta \frac{d\theta}{dt}.$$

When $\theta = \pi/6$, $\sin^2 \theta = 1/4$, so $\csc^2 \theta = 4$. Converting $d\theta/dt = 1/3 \text{ deg/s}$ to radians per hour, we find

$$\frac{d\theta}{dt} = \frac{1}{3} \left(\frac{\pi}{180} \right) (3600) \text{ rad/h.} \quad 1 \text{ h} = 3600 \text{ s, } 1 \text{ deg} = \pi/180 \text{ rad}$$

Substitution into the equation for dx/dt then gives

$$\frac{dx}{dt} = (-10)(4) \left(\frac{1}{3} \right) \left(\frac{\pi}{180} \right) (3600) \approx -838.$$

The negative sign appears because the distance x is decreasing, so the aircraft is approaching the island at a speed of approximately 838 km/h when first detected by the radar. ■

Note that the solution of Example 5 involved several unit conversions: from seconds to hours and from degrees to radians. When solving related rates problems, we should check that consistent units are used.

EXAMPLE 6 Figure 3.50a shows a rope running through a pulley at P and bearing a weight W at one end. The other end is held 1.5 m above the ground in the hand M of a worker. Suppose the pulley is 7.5 m above ground, the rope is 13.5 m long, and the worker is walking rapidly away from the vertical line PW at the rate of 1.2 m/s. How fast is the weight being raised when the worker's hand is 6.3 m away from PW ?

Solution We let OM be the horizontal line of length x m from a point O directly below the pulley to the worker's hand M at any instant of time (Figure 3.50). Let h be the height of the weight W above O , and let z denote the length of rope from the pulley P to the worker's hand. We want to know dh/dt when $x = 6.3$ given that $dx/dt = 1.2$. Note that the height of P above O is 6 m because O is 1.5 m above the ground. We assume the angle at O is a right angle.

At any instant of time t , we have the following relationships (see Figure 3.50b):

$$6 - h + z = 13.5 \quad \text{Total length of rope is 13.5 m.}$$

$$6^2 + x^2 = z^2. \quad \text{Angle at } O \text{ is a right angle.}$$

If we solve for $z = 7.5 + h$ in the first equation, and substitute into the second equation, we have

$$6^2 + x^2 = (7.5 + h)^2. \quad (1)$$

Differentiating both sides with respect to t gives

$$2x \frac{dx}{dt} = 2(7.5 + h) \frac{dh}{dt},$$

and solving this last equation for dh/dt we find

$$\frac{dh}{dt} = \frac{x}{7.5 + h} \frac{dx}{dt}. \quad (2)$$

Since we know dx/dt , it remains only to find $7.5 + h$ at the instant when $x = 6.3$. From Equation (1),

$$6^2 + 6.3^2 = (7.5 + h)^2$$

so that

$$(7.5 + h)^2 = 75.69, \text{ or } 7.5 + h = 8.7.$$

Equation (2) now gives

$$\frac{dh}{dt} = \frac{6.3}{8.7} \cdot 1.2 = \frac{756}{870} \approx 0.87 \text{ m/s}$$

as the rate at which the weight is being raised when $x = 6.3$ m. ■

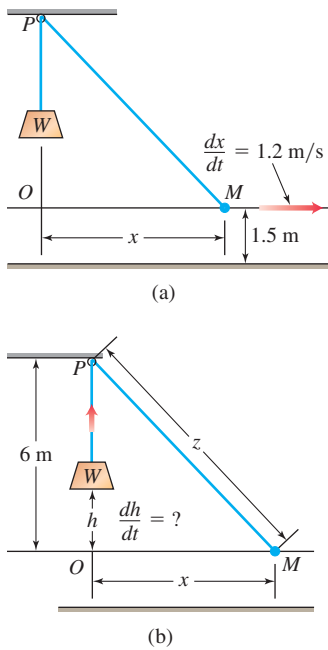


FIGURE 3.50 A worker at M walks to the right, pulling the weight W upward as the rope moves through the pulley P (Example 6).

EXERCISES 3.10

- Area** Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t . Write an equation that relates dA/dt to dr/dt .
- Surface area** Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t . Write an equation that relates dS/dt to dr/dt .
- Assume that $y = 5x$ and $dx/dt = 2$. Find dy/dt .
- Assume that $2x + 3y = 12$ and $dy/dt = -2$. Find dx/dt .
- If $y = x^2$ and $dx/dt = 3$, then what is dy/dt when $x = -1$?
- If $x = y^3 - y$ and $dy/dt = 5$, then what is dx/dt when $y = 2$?
- If $x^2 + y^2 = 25$ and $dx/dt = -2$, then what is dy/dt when $x = 3$ and $y = -4$?
- If $x^2y^3 = 4/27$ and $dy/dt = 1/2$, then what is dx/dt when $x = 2$?
- If $L = \sqrt{x^2 + y^2}$, $dx/dt = -1$, and $dy/dt = 3$, find dL/dt when $x = 5$ and $y = 12$.
- If $r + s^2 + v^3 = 12$, $dr/dt = 4$, and $ds/dt = -3$, find dv/dt when $r = 3$ and $s = 1$.
- If the original 24 m edge length x of a cube decreases at the rate of 5 m/min, when $x = 3$ m at what rate does the cube's
 - surface area change?
 - volume change?
- A cube's surface area increases at the rate of $72 \text{ cm}^2/\text{s}$. At what rate is the cube's volume changing when the edge length is $x = 3$ cm?
- Volume** The radius r and height h of a right circular cylinder are related to the cylinder's volume V by the formula $V = \pi r^2 h$.
 - How is dV/dt related to dh/dt if r is constant?
 - How is dV/dt related to dr/dt if h is constant?
 - How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?
- Volume** The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = (1/3)\pi r^2 h$.
 - How is dV/dt related to dh/dt if r is constant?
 - How is dV/dt related to dr/dt if h is constant?
 - How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?
- Changing voltage** The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/s while I is decreasing at the rate of $1/3$ amp/s. Let t denote time in seconds.
- Electrical power** The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.
 - How are dP/dt , dR/dt , and dI/dt related if none of P , R , and I are constant?
 - How is dR/dt related to dI/dt if P is constant?
- Distance** Let x and y be differentiable functions of t , and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy -plane.
 - How is ds/dt related to dx/dt if y is constant?
 - How is ds/dt related to dx/dt and dy/dt if neither x nor y is constant?
 - How is dx/dt related to dy/dt if s is constant?
- Diagonals** If x , y , and z are lengths of the edges of a rectangular box, then the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$.
 - Assuming that x , y , and z are differentiable functions of t , how is ds/dt related to dx/dt , dy/dt , and dz/dt ?
 - How is ds/dt related to dy/dt and dz/dt if x is constant?
 - How are dx/dt , dy/dt , and dz/dt related if s is constant?
- Area** The area A of a triangle with sides of lengths a and b enclosing an angle of measure θ is

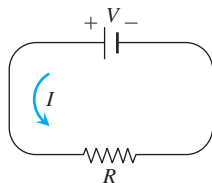
$$A = \frac{1}{2}ab \sin \theta.$$

- How is dA/dt related to $d\theta/dt$ if a and b are constant?
 - How is dA/dt related to $d\theta/dt$ and da/dt if only b is constant?
 - How is dA/dt related to $d\theta/dt$, da/dt , and db/dt if none of a , b , and θ are constant?
- Heating a plate** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min . At what rate is the plate's area increasing when the radius is 50 cm?
 - Changing dimensions in a rectangle** The length l of a rectangle is decreasing at the rate of 2 cm/s while the width w is increasing at the rate of 2 cm/s . When $l = 12 \text{ cm}$ and $w = 5 \text{ cm}$, find the rates of change of (a) the area, (b) the perimeter, and (c) the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?
 - Changing dimensions in a rectangular box** Suppose that the edge lengths x , y , and z of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/s}, \quad \frac{dy}{dt} = -2 \text{ m/s}, \quad \frac{dz}{dt} = 1 \text{ m/s}.$$

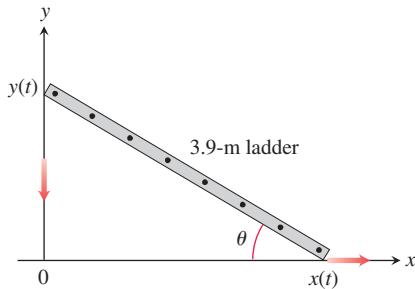
Find the rates at which the box's (a) volume, (b) surface area, and (c) diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

- A sliding ladder** A 3.9-m ladder is leaning against a house when its base starts to slide away (see accompanying figure). By the time the base is 3.6 m from the house, the base is moving at the rate of 1.5 m/s .
 - How fast is the top of the ladder sliding down the wall then?

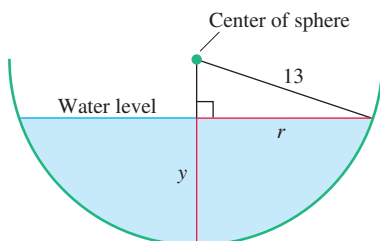


- What is the value of dV/dt ?
- What is the value of dI/dt ?
- What equation relates dR/dt to dV/dt and dI/dt ?
- Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amps. Is R increasing, or decreasing?

- b. At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
- c. At what rate is the angle θ between the ladder and the ground changing then?

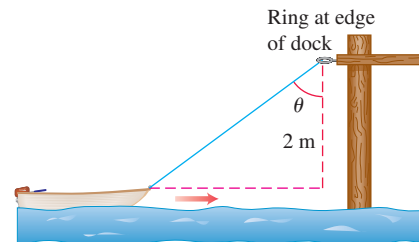


24. **Commercial air traffic** Two commercial airplanes are flying at an altitude of 12,000 m along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 1852 m). Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when A is 5 nautical miles from the intersection point, and B is 12 nautical miles from the intersection point?
25. **Flying a kite** A girl flies a kite at a height of 90 m, the wind carrying the kite horizontally away from her at a rate of 7.5 m/s. How fast must she let out the string when the kite is 150 m away from her?
26. **Boring a cylinder** The mechanics at Lincoln Automotive are re boring a 15-cm-deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of a centimeter every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 10 cm?
27. **A growing sand pile** Sand falls from a conveyor belt at the rate of $10 \text{ m}^3/\text{min}$ onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the (a) height and (b) radius changing when the pile is 4 m high? Answer in centimeters per minute.
28. **A draining conical reservoir** Water is flowing at the rate of $50 \text{ m}^3/\text{min}$ from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.
- a. How fast (in centimeters per minute) is the water level falling when the water is 5 m deep?
- b. How fast is the radius of the water's surface changing then? Answer in centimeters per minute.
29. **A draining hemispherical reservoir** Water is flowing at the rate of $6 \text{ m}^3/\text{min}$ from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y meters deep.

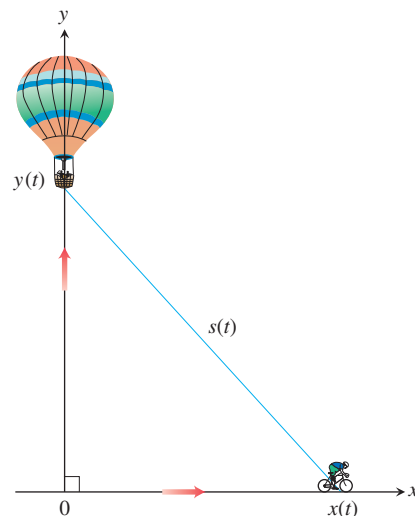


- a. At what rate is the water level changing when the water is 8 m deep?
- b. What is the radius r of the water's surface when the water is y m deep?
- c. At what rate is the radius r changing when the water is 8 m deep?

30. **A growing raindrop** Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.
31. **The radius of an inflating balloon** A spherical balloon is inflated with helium at the rate of $100\pi \text{ m}^3/\text{min}$. How fast is the balloon's radius increasing at the instant the radius is 5 m? How fast is the surface area increasing?
32. **Hauling in a dinghy** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 2 m above the bow. The rope is hauled in at the rate of 0.5 m/s.
- a. How fast is the boat approaching the dock when 3 m of rope are out?
- b. At what rate is the angle θ changing at this instant (see the figure)?

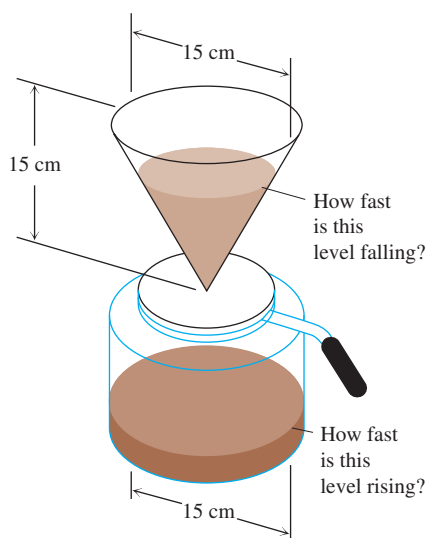


33. **A balloon and a bicycle** A balloon is rising vertically above a level, straight road at a constant rate of 0.3 m/s. Just when the balloon is 20 m above the ground, a bicycle moving at a constant rate of 5 m/s passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 s later?



34. **Making coffee** Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of $160 \text{ cm}^3/\text{min}$.
- a. How fast is the level in the pot rising when the coffee in the cone is 12 cm deep?

b. How fast is the level in the cone falling then?



35. Cardiac output In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würzburg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 L/min. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (ml/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ ml/min and $D = 97 - 56 = 41$ ml/L,

$$y = \frac{233 \text{ ml/min}}{41 \text{ ml/L}} \approx 5.68 \text{ L/min},$$

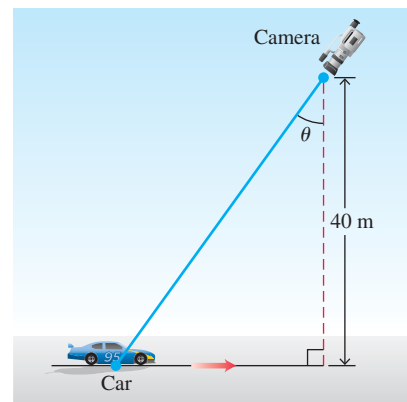
fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

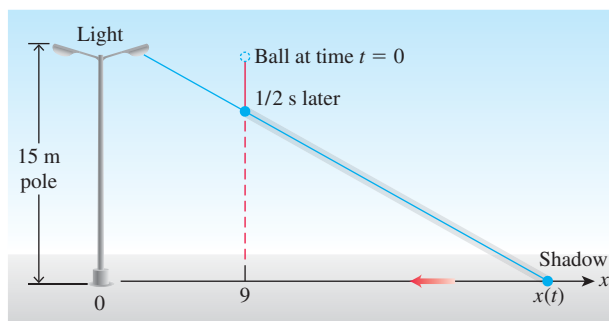
36. Moving along a parabola A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x -coordinate (measured in meters) increases at a steady 10 m/s. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m?

37. Motion in the plane The coordinates of a particle in the metric xy -plane are differentiable functions of time t with $dx/dt = -1$ m/s and $dy/dt = -5$ m/s. How fast is the particle's distance from the origin changing as it passes through the point $(5, 12)$?

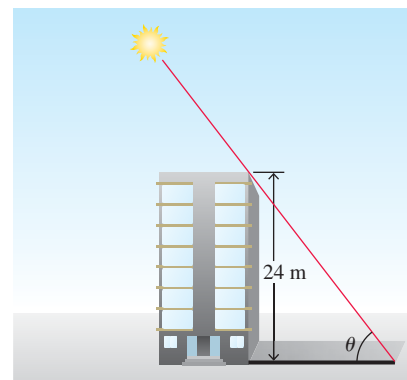
38. Videotaping a moving car You are videotaping a race from a stand 40 m from the track, following a car that is moving at 288 km/h (80 m/s), as shown in the accompanying figure. How fast will your camera angle θ be changing when the car is right in front of you? A half second later?



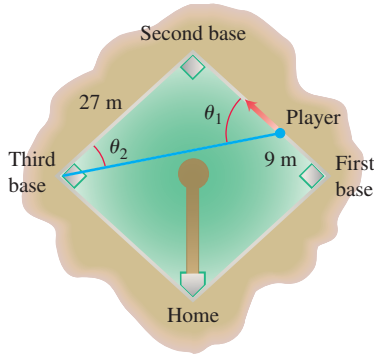
39. A moving shadow A light shines from the top of a pole 15 m high. A ball is dropped from the same height from a point 9 m away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground $1/2$ s later? (Assume the ball falls a distance $s = 4.9t^2$ m in t seconds.)



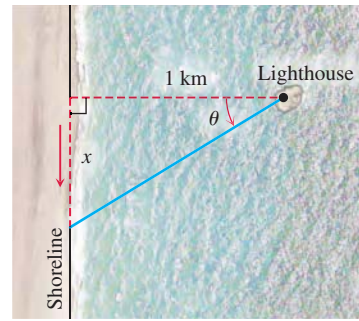
40. A building's shadow On a morning of a day when the sun will pass directly overhead, the shadow of a 24 m building on level ground is 18 m long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of $0.27^\circ/\text{min}$. At what rate is the shadow decreasing? (Remember to use radians. Express your answer in centimeters per minute, to the nearest tenth.)



- 41. A melting ice layer** A spherical iron ball 8 cm in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of $10 \text{ cm}^3/\text{min}$, how fast is the thickness of the ice decreasing when it is 2 cm thick? How fast is the outer surface area of ice decreasing?
- 42. Highway patrol** A highway patrol plane flies 3 km above a level, straight road at a steady 120 km/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 km, the line-of-sight distance is decreasing at the rate of 160 km/h. Find the car's speed along the highway.
- 43. Baseball players** A baseball diamond is a square 27 m on a side. A player runs from first base to second at a rate of 5 m/s.
- At what rate is the player's distance from third base changing when the player is 9 m from first base?
 - At what rates are angles θ_1 and θ_2 (see the figure) changing at that time?
 - The player slides into second base at the rate of 4.5 m/s. At what rates are angles θ_1 and θ_2 changing as the player touches base?



- 44. Ships** Two ships are steaming straight away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 1852 m). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?
- 45. Clock's moving hands** At what rate is the angle between a clock's minute and hour hands changing at 4 o'clock in the afternoon?
- 46. Oil spill** An explosion at an oil rig located in gulf waters causes an elliptical oil slick to spread on the surface from the rig. The slick is a constant 20 cm thick. After several days, when the major axis of the slick is 2 km long and the minor axis is $3/4$ km wide, it is determined that its length is increasing at the rate of 9 m/h, and its width is increasing at the rate of 3 m/h. At what rate (in cubic meters per hour) is oil flowing from the site of the rig at that time?
- 47. A lighthouse beam** A lighthouse sits 1 km offshore, and its beam of light rotates counterclockwise at the constant rate of 3 full circles per minute. At what rate is the image of the beam moving down the shoreline when the image is 1 km from the spot on the shoreline nearest the lighthouse?



3.11 Linearization and Differentials

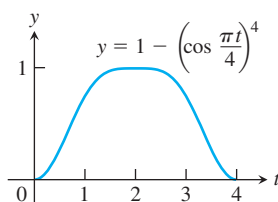
It is often useful to approximate complicated functions with simpler ones that give the accuracy we want for specific applications and, at the same time, are easier to work with than the original functions. The approximating functions discussed in this section are called *linearizations*, and they are based on tangent lines. Other approximating functions, such as polynomials, are discussed in Chapter 9.

We introduce new variables dx and dy , called *differentials*, and define them in a way that makes Leibniz's notation for the derivative dy/dx a true ratio. We use dy to estimate error in measurement, which then provides for a precise proof of the Chain Rule (Section 3.6).

Linearization

As you can see in Figure 3.51, the tangent line to the curve $y = x^2$ lies close to the curve near the point of tangency. For a brief interval to either side, the y -values along the tangent line give good approximations to the y -values on the curve. We observe this phenomenon

T 18. $f(t) = 1 - \left(\cos \frac{\pi t}{4}\right)^4$ on $[0, 4]$



Estimations

19. Water pollution Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

Time (h)	0	1	2	3	4
Leakage (L/h)	50	70	97	136	190
Time (h)	5	6	7	8	
Leakage (L/h)	265	369	516	720	

- Give an upper and a lower estimate of the total quantity of oil that has escaped after 5 hours.
 - Repeat part (a) for the quantity of oil that has escaped after 8 hours.
 - The tanker continues to leak 720 L/h after the first 8 hours. If the tanker originally contained 25,000 L of oil, approximately how many more hours will elapse in the worst case before all the oil has spilled? In the best case?
- 20. Air pollution** A power plant generates electricity by burning oil. Pollutants produced as a result of the burning process are removed by scrubbers in the smokestacks. Over time, the scrubbers become less efficient and eventually they must be replaced when the amount of pollution released exceeds government standards. Measurements are taken at the end of each month determining the rate at which pollutants are released into the atmosphere, recorded as follows.

Month	Jan	Feb	Mar	Apr	May	Jun
Pollutant release rate (tons/day)	0.20	0.25	0.27	0.34	0.45	0.52

Month	Jul	Aug	Sep	Oct	Nov	Dec
Pollutant release rate (tons/day)	0.63	0.70	0.81	0.85	0.89	0.95

- Assuming a 30-day month and that new scrubbers allow only 0.05 ton/day to be released, give an upper estimate of the total tonnage of pollutants released by the end of June. What is a lower estimate?
- In the best case, approximately when will a total of 125 tons of pollutants have been released into the atmosphere?

T 21. Inscribe a regular n -sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of n :

- 4 (square)
- 8 (octagon)
- 16
- Compare the areas in parts (a), (b), and (c) with the area inside the circle.

22. (Continuation of Exercise 21.)

- Inscribe a regular n -sided polygon inside a circle of radius 1 and compute the area of one of the n congruent triangles formed by drawing radii to the vertices of the polygon.
- Compute the limit of the area of the inscribed polygon as $n \rightarrow \infty$.
- Repeat the computations in parts (a) and (b) for a circle of radius r .

COMPUTER EXPLORATIONS

In Exercises 23–26, use a CAS to perform the following steps.

- Plot the functions over the given interval.
- Subdivide the interval into $n = 100, 200$, and 1000 subintervals of equal length and evaluate the function at the midpoint of each subinterval.
- Compute the average value of the function values generated in part (b).
- Solve the equation $f(x) = (\text{average value})$ for x using the average value calculated in part (c) for the $n = 1000$ partitioning.

23. $f(x) = \sin x$ on $[0, \pi]$

24. $f(x) = \sin^2 x$ on $[0, \pi]$

25. $f(x) = x \sin \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

26. $f(x) = x \sin^2 \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

5.2 Sigma Notation and Limits of Finite Sums

While estimating with finite sums in Section 5.1, we encountered sums that had many terms (up to 1000 terms in Table 5.1). In this section we introduce a notation for sums that have a large number of terms. After describing this notation and its properties, we consider what happens as the number of terms in a sum approaches infinity.

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

Σ is the capital Greek letter sigma

The Greek letter Σ (capital sigma, corresponding to our letter S), stands for “sum.” The **index of summation** k tells us where the sum begins (at the number below the Σ symbol) and where it ends (at the number above Σ). Any letter can be used to denote the index, but the letters i , j , k , and n are customary.

$$\sum_{k=1}^n a_k$$

The summation symbol (Greek letter sigma) The index k ends at $k = n$.
 a_k is a formula for the k th term.
The index k starts at $k = 1$.

Thus we can write the sum of the squares of the numbers 1 through 11 as

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2,$$

and the sum of $f(i)$ for integers i from 1 to 100 as

$$f(1) + f(2) + f(3) + \cdots + f(100) = \sum_{i=1}^{100} f(i).$$

The starting index does not have to be 1; it can be any integer.

EXAMPLE 1

A sum in sigma notation	The sum written out, one term for each value of k	The value of the sum
$\sum_{k=1}^5 k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^3 (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^2 \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^5 \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

EXAMPLE 2 Express the sum $1 + 3 + 5 + 7 + 9$ in sigma notation.

Solution The formula generating the terms depends on what we choose the lower limit of summation to be, but the terms generated remain the same. It is often simplest to choose the starting index to be $k = 0$ or $k = 1$, but we can start with any integer.

$$\text{Starting with } k = 0: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=0}^4 (2k + 1)$$

$$\text{Starting with } k = 1: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=1}^5 (2k - 1)$$

$$\text{Starting with } k = 2: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=2}^6 (2k - 3)$$

$$\text{Starting with } k = -3: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=-3}^1 (2k + 7)$$

When we have a sum such as

$$\sum_{k=1}^3 (k + k^2),$$

we can rearrange its terms to form two sums:

$$\begin{aligned}\sum_{k=1}^3 (k + k^2) &= (1 + 1^2) + (2 + 2^2) + (3 + 3^2) \\ &= (1 + 2 + 3) + (1^2 + 2^2 + 3^2) \quad \text{Regroup terms.} \\ &= \sum_{k=1}^3 k + \sum_{k=1}^3 k^2.\end{aligned}$$

This illustrates a general rule for finite sums:

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k.$$

This and three other rules are given below. Proofs of these rules can be obtained using mathematical induction (see Appendix A.3).

Algebra Rules for Finite Sums

1. *Sum Rule:* $\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$
2. *Difference Rule:* $\sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n a_k - \sum_{k=1}^n b_k$
3. *Constant Multiple Rule:* $\sum_{k=1}^n c a_k = c \cdot \sum_{k=1}^n a_k$ (Any number c)
4. *Constant Value Rule:* $\sum_{k=1}^n c = n \cdot c$ (Any number c)

EXAMPLE 3 We demonstrate the use of the algebra rules.

- (a) $\sum_{k=1}^n (3k - k^2) = 3 \sum_{k=1}^n k - \sum_{k=1}^n k^2$ Difference Rule and Constant Multiple Rule
- (b) $\sum_{k=1}^n (-a_k) = \sum_{k=1}^n (-1) \cdot a_k = -1 \cdot \sum_{k=1}^n a_k = -\sum_{k=1}^n a_k$ Constant Multiple Rule
- (c) $\sum_{k=1}^3 (k + 4) = \sum_{k=1}^3 k + \sum_{k=1}^3 4$ Sum Rule
 $= (1 + 2 + 3) + (3 \cdot 4)$ Constant Value Rule
 $= 6 + 12 = 18$
- (d) $\sum_{k=1}^n \frac{1}{n} = n \cdot \frac{1}{n} = 1$ Constant Value Rule
($1/n$ is constant)

HISTORICAL BIOGRAPHY

Carl Friedrich Gauss
(1777–1855)

Gauss was born in Brunswick, Germany. The list of Gauss's accomplishments in science and mathematics is astonishing, ranging from the invention of the electric telegraph (with Wilhelm Weber in 1833) to the development of a theory of planetary orbits and the development of an accurate theory of non-Euclidean geometry.

To know more, visit the companion Website.

Over the years, people have discovered a variety of formulas for the values of finite sums. The most famous of these are the formula for the sum of the first n positive integers (Gauss is said to have discovered it at age 8) and the formulas for the sums of the squares and cubes of the first n positive integers.

EXAMPLE 4 Show that the sum of the first n positive integers is

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

Solution The formula tells us that the sum of the first 4 positive integers is

$$\frac{(4)(5)}{2} = 10.$$

Addition verifies this prediction:

$$1 + 2 + 3 + 4 = 10.$$

To prove the formula in general, we write out the terms in the sum twice, once forward and once backward.

$$\begin{array}{ccccccc} 1 & + & 2 & + & 3 & + & \cdots + n \\ n & + & (n-1) & + & (n-2) & + & \cdots + 1 \end{array}$$

If we add the two terms in the first column we get $1 + n = n + 1$. Similarly, if we add the two terms in the second column we get $2 + (n - 1) = n + 1$. The two terms in any column sum to $n + 1$. When we add the n columns together we get n terms, each equal to $n + 1$, for a total of $n(n + 1)$. Since this is twice the desired quantity, the sum of the first n integers is $n(n + 1)/2$. ■

Formulas for the sums of the squares and cubes of the first n positive integers are proved using mathematical induction (see Appendix A.3). We state them here.

Sum of the first n squares: $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$ Sum of the first n cubes: $\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Limits of Finite Sums

The finite sum approximations that we considered in Section 5.1 became more accurate as the number of terms increased and the subinterval widths (lengths) narrowed. The next example shows how to calculate a limiting value as the widths of the subintervals go to zero and the number of subintervals grows to infinity.

EXAMPLE 5 Find the limiting value of lower sum approximations to the area of the region R below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$ on the x -axis using equal-width rectangles whose widths approach zero and whose number approaches infinity. (See Figure 5.4a.)

Solution We compute a lower sum approximation using n rectangles of equal width Δx and then see what happens as $n \rightarrow \infty$. We start by subdividing $[0, 1]$ into n equal width subintervals

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \dots, \left[\frac{n-1}{n}, \frac{n}{n}\right].$$

Each subinterval has width $\Delta x = 1/n$. The function $1 - x^2$ is decreasing on $[0, 1]$, and its smallest value in a subinterval occurs at the subinterval's right endpoint. So a lower sum is constructed with rectangles whose height over the subinterval $[(k-1)/n, k/n]$ is $f(k/n) = 1 - (k/n)^2$, giving the sum

$$f\left(\frac{1}{n}\right) \cdot \frac{1}{n} + f\left(\frac{2}{n}\right) \cdot \frac{1}{n} + \cdots + f\left(\frac{k}{n}\right) \cdot \frac{1}{n} + \cdots + f\left(\frac{n}{n}\right) \cdot \frac{1}{n}.$$

We write this in sigma notation and simplify,

$$\begin{aligned}
 \sum_{k=1}^n f\left(\frac{k}{n}\right) \cdot \frac{1}{n} &= \sum_{k=1}^n \left(1 - \left(\frac{k}{n}\right)^2\right) \frac{1}{n} \\
 &= \sum_{k=1}^n \left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
 &= \sum_{k=1}^n \frac{1}{n} - \sum_{k=1}^n \frac{k^2}{n^3} && \text{Difference Rule} \\
 &= n \cdot \frac{1}{n} - \frac{1}{n^3} \sum_{k=1}^n k^2 && \text{Constant Value and Constant Multiple Rules} \\
 &= 1 - \left(\frac{1}{n^3}\right) \frac{n(n+1)(2n+1)}{6} && \text{Sum of the first } n \text{ squares} \\
 &= 1 - \frac{2n^3 + 3n^2 + n}{6n^3}. && \text{Numerator expanded}
 \end{aligned}$$

We have obtained an expression for the lower sum that holds for any n . Taking the limit of this expression as $n \rightarrow \infty$, we see that the lower sums converge as the number of subintervals increases and the subinterval widths approach zero:

$$\lim_{n \rightarrow \infty} \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3}\right) = 1 - \frac{2}{6} = \frac{2}{3}.$$

The lower sum approximations converge to $2/3$. A similar calculation shows that the upper sum approximations also converge to $2/3$. Any finite sum approximation $\sum_{k=1}^n f(c_k)(1/n)$ also converges to the same value, $2/3$. This is because it is possible to show that any finite sum approximation is trapped between the lower and upper sum approximations. For this reason we are led to *define* the area of the region R as this limiting value. In Section 5.3 we study the limits of such finite approximations in a general setting. ■

HISTORICAL BIOGRAPHY

Georg Friedrich Bernhard Riemann (1826–1866)

Riemann was born in Hanover, Germany. His doctorate was obtained under the direction of Gauss in the theory of complex variables. He also worked with physicist Wilhelm Weber. He introduced the foundational ideas of differential geometry and contributed to dynamics, non-Euclidean geometry, and computational physics.

To know more, visit the companion Website.

Riemann Sums

The theory of limits of finite approximations was made precise by the German mathematician Bernhard Riemann. We now introduce the notion of a *Riemann sum*, which underlies the theory of the definite integral that will be presented in the next section.

We begin with an arbitrary bounded function f defined on a closed interval $[a, b]$. Like the function pictured in Figure 5.8, f may have negative as well as positive values. We subdivide the interval $[a, b]$ into subintervals, not necessarily of equal width (or length), and form sums in the same way as for the finite approximations in Section 5.1. To do so, we choose $n - 1$ points $\{x_1, x_2, x_3, \dots, x_{n-1}\}$ between a and b that are in increasing order, so that

$$a < x_1 < x_2 < \dots < x_{n-1} < b.$$

To make the notation consistent, we set $x_0 = a$ and $x_n = b$, so that

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b.$$

The set of all of these points,

$$P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\},$$

is called a **partition** of $[a, b]$.

The partition P divides $[a, b]$ into the n closed subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n].$$

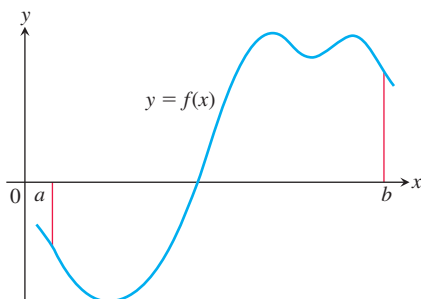


FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

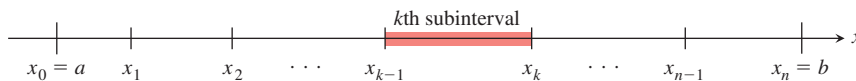
HISTORICAL BIOGRAPHY

Richard Dedekind
(1831–1916)

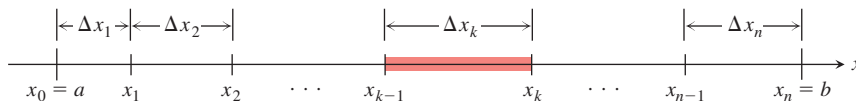
Dedekind grew up in Germany and in 1850 entered the University of Göttingen. There he studied with Bernhard Riemann and Carl Gauss. Like Gauss, Dedekind preferred to study the theoretical aspects of number theory. His work on irrational numbers gave the subject a logical foundation.

To know more, visit the companion Website.

The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the k th subinterval is $[x_{k-1}, x_k]$ (where k is an integer between 1 and n).



The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1 , the width of the second $[x_1, x_2]$ is Δx_2 , and the width of the k th subinterval is $\Delta x_k = x_k - x_{k-1}$.



If all n subintervals have equal width, then their common width, which we call Δx , is equal to $(b - a)/n$. Using equal width subintervals is often the simplest choice when doing computations.

In each subinterval we select some point. The point chosen in the k th subinterval $[x_{k-1}, x_k]$ is called c_k . Then on each subinterval, we stand a vertical rectangle that stretches from the x -axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x -axis, depending on whether $f(c_k)$ is positive or negative, or on the x -axis if $f(c_k) = 0$ (see Figure 5.9).

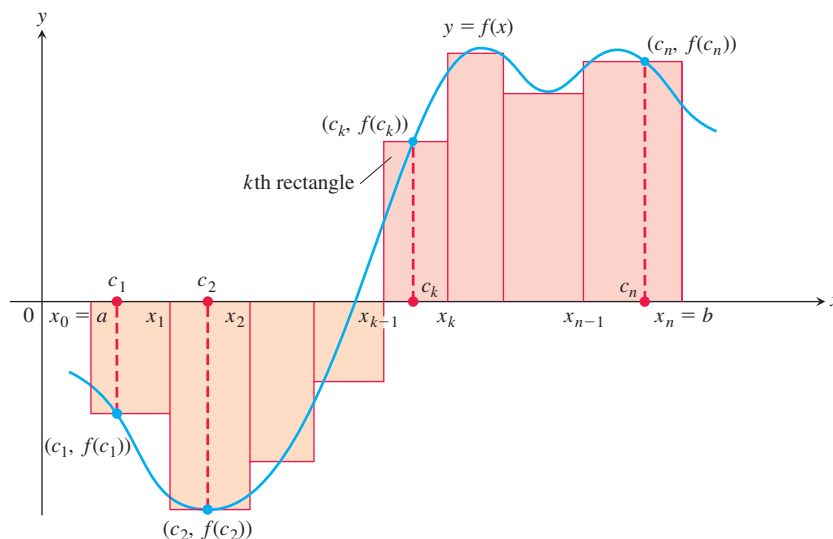


FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x -axis. Figure 5.8 has been repeated and enlarged, the partition of $[a, b]$ and the points c_k have been added, and the corresponding rectangles with heights $f(c_k)$ are shown.

On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k . When $f(c_k) < 0$, the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x -axis to the negative number $f(c_k)$.

Finally, we sum all these products to get

$$S_P = \sum_{k=1}^n f(c_k) \Delta x_k.$$

The sum S_P is called a **Riemann sum for f on the interval $[a, b]$** . There are many such sums, depending on the partition P we choose and on the choices of the points c_k in the

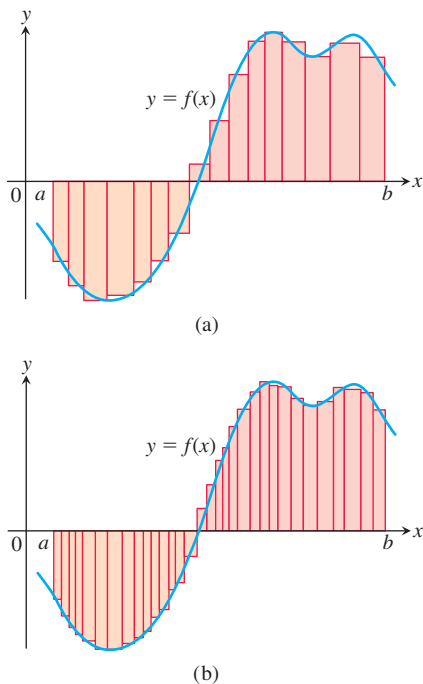


FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x -axis with increasing accuracy.

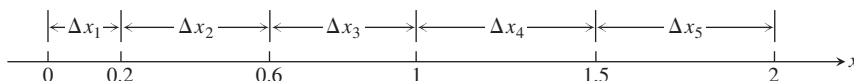
subintervals. For instance, we could choose n subintervals all having equal width $\Delta x = (b - a)/n$ to partition $[a, b]$, and then choose the point c_k to be the right-hand endpoint of each subinterval when forming the Riemann sum (as we did in Example 5). This choice leads to the Riemann sum formula

$$S_n = \sum_{k=1}^n f\left(a + k \frac{b-a}{n}\right) \cdot \left(\frac{b-a}{n}\right).$$

Similar formulas can be obtained if instead we choose c_k to be the left-hand endpoint, or the midpoint, of each subinterval.

In the cases in which the subintervals all have equal width $\Delta x = (b - a)/n$, we can make them thinner by simply increasing their number n . When a partition has subintervals of varying widths, we can ensure they are all thin by controlling the width of a widest (longest) subinterval. We define the **norm** of a partition P , written $\|P\|$, to be the largest of all the subinterval widths. If $\|P\|$ is a small number, then all of the subintervals in the partition P have a small width.

EXAMPLE 6 The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$. There are five subintervals of P : $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$:



The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$. The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length. ■

Any Riemann sum associated with a partition of a closed interval $[a, b]$ defines rectangles that approximate the region between the graph of a continuous function f and the x -axis. Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy, as suggested by Figure 5.10. We will see in the next section that if the function f is continuous over the closed interval $[a, b]$, then no matter how we choose the partition P and the points c_k in its subintervals, the Riemann sums corresponding to these choices will approach a single limiting value as the subinterval widths (which are controlled by the norm of the partition) approach zero.

EXERCISES 5.2

Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

1. $\sum_{k=1}^2 \frac{6k}{k+1}$

2. $\sum_{k=1}^3 \frac{k-1}{k}$

3. $\sum_{k=1}^4 \cos k\pi$

4. $\sum_{k=1}^5 \sin k\pi$

5. $\sum_{k=1}^3 (-1)^{k+1} \sin \frac{\pi}{k}$

6. $\sum_{k=1}^4 (-1)^k \cos k\pi$

7. Which of the following express $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation?

a. $\sum_{k=1}^6 2^{k-1}$

b. $\sum_{k=0}^5 2^k$

c. $\sum_{k=-1}^4 2^{k+1}$

8. Which of the following express $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation?

a. $\sum_{k=1}^6 (-2)^{k-1}$

b. $\sum_{k=0}^5 (-1)^k 2^k$

c. $\sum_{k=-2}^3 (-1)^{k+1} 2^{k+2}$

9. Which formula is not equivalent to the other two?

a. $\sum_{k=2}^4 \frac{(-1)^{k-1}}{k-1}$

b. $\sum_{k=0}^2 \frac{(-1)^k}{k+1}$

c. $\sum_{k=-1}^1 \frac{(-1)^k}{k+2}$

10. Which formula is not equivalent to the other two?

a. $\sum_{k=1}^4 (k-1)^2$

b. $\sum_{k=-1}^3 (k+1)^2$

c. $\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice for the starting index.

11. $1 + 2 + 3 + 4 + 5 + 6$

12. $1 + 4 + 9 + 16$

13. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

14. $2 + 4 + 6 + 8 + 10$

$$15. 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \qquad 16. -\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$$

Values of Finite Sums

17. Suppose that $\sum_{k=1}^n a_k = -5$ and $\sum_{k=1}^n b_k = 6$. Find the values of

$$\begin{array}{lll} \text{a. } \sum_{k=1}^n 3a_k & \text{b. } \sum_{k=1}^n \frac{b_k}{6} & \text{c. } \sum_{k=1}^n (a_k + b_k) \\ \text{d. } \sum_{k=1}^n (a_k - b_k) & \text{e. } \sum_{k=1}^n (b_k - 2a_k) & \end{array}$$

18. Suppose that $\sum_{k=1}^n a_k = 0$ and $\sum_{k=1}^n b_k = 1$. Find the values of

$$\begin{array}{ll} \text{a. } \sum_{k=1}^n 8a_k & \text{b. } \sum_{k=1}^n 250b_k \\ \text{c. } \sum_{k=1}^n (a_k + 1) & \text{d. } \sum_{k=1}^n (b_k - 1) \end{array}$$

Evaluate the sums in Exercises 19–36.

$$19. \text{ a. } \sum_{k=1}^{10} k \qquad \text{b. } \sum_{k=1}^{10} k^2 \qquad \text{c. } \sum_{k=1}^{10} k^3$$

$$20. \text{ a. } \sum_{k=1}^{13} k \qquad \text{b. } \sum_{k=1}^{13} k^2 \qquad \text{c. } \sum_{k=1}^{13} k^3$$

$$21. \sum_{k=1}^7 (-2k) \qquad 22. \sum_{k=1}^5 \frac{\pi k}{15}$$

$$23. \sum_{k=1}^6 (3 - k^2) \qquad 24. \sum_{k=1}^6 (k^2 - 5)$$

$$25. \sum_{k=1}^5 k(3k + 5) \qquad 26. \sum_{k=1}^7 k(2k + 1)$$

$$27. \sum_{k=1}^5 \frac{k^3}{225} + \left(\sum_{k=1}^5 k \right)^3 \qquad 28. \left(\sum_{k=1}^7 k \right)^2 - \sum_{k=1}^7 \frac{k^3}{4}$$

$$29. \text{ a. } \sum_{k=1}^7 3 \qquad \text{b. } \sum_{k=1}^{500} 7 \qquad \text{c. } \sum_{k=3}^{264} 10$$

$$30. \text{ a. } \sum_{k=9}^{36} k \qquad \text{b. } \sum_{k=3}^{17} k^2 \qquad \text{c. } \sum_{k=18}^{71} k(k - 1)$$

$$31. \text{ a. } \sum_{k=1}^n 4 \qquad \text{b. } \sum_{k=1}^n c \qquad \text{c. } \sum_{k=1}^n (k - 1)$$

$$32. \text{ a. } \sum_{k=1}^n \left(\frac{1}{n} + 2n \right) \qquad \text{b. } \sum_{k=1}^n \frac{c}{n} \qquad \text{c. } \sum_{k=1}^n \frac{k}{n^2}$$

$$33. \sum_{k=1}^{50} [(k + 1)^2 - k^2] \qquad 34. \sum_{k=2}^{20} [\sin(k - 1) - \sin k]$$

$$35. \sum_{k=7}^{30} (\sqrt{k - 4} - \sqrt{k - 3})$$

$$36. \sum_{k=1}^{40} \frac{1}{k(k + 1)} \quad \left(\text{Hint: } \frac{1}{k(k + 1)} = \frac{1}{k} - \frac{1}{k + 1} \right)$$

Riemann Sums

In Exercises 37–40, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^4 f(c_k) \Delta x_k$, given that c_k is the (a) left-hand endpoint, (b) right-hand endpoint, (c) midpoint of the k th subinterval. (Make a separate sketch for each set of rectangles.)

$$37. f(x) = x^2 - 1, \quad [0, 2] \qquad 38. f(x) = -x^2, \quad [0, 1]$$

$$39. f(x) = \sin x, \quad [-\pi, \pi]$$

$$40. f(x) = \sin x + 1, \quad [-\pi, \pi]$$

41. Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.

42. Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

Limits of Riemann Sums

For the functions in Exercises 43–50, find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into n equal subintervals and using the right-hand endpoint for each c_k . Then take a limit of these sums as $n \rightarrow \infty$ to calculate the area under the curve over $[a, b]$.

$$43. f(x) = 1 - x^2 \text{ over the interval } [0, 1].$$

$$44. f(x) = 2x \text{ over the interval } [0, 3].$$

$$45. f(x) = x^2 + 1 \text{ over the interval } [0, 3].$$

$$46. f(x) = 3x^2 \text{ over the interval } [0, 1].$$

$$47. f(x) = x + x^2 \text{ over the interval } [0, 1].$$

$$48. f(x) = 3x + 2x^2 \text{ over the interval } [0, 1].$$

$$49. f(x) = 2x^3 \text{ over the interval } [0, 1].$$

$$50. f(x) = x^2 - x^3 \text{ over the interval } [-1, 0].$$

5.3 The Definite Integral

In this section we consider the limit of general Riemann sums as the norm of the partitions of a closed interval $[a, b]$ approaches zero. This limiting process leads us to the definition of the *definite integral* of a function over a closed interval $[a, b]$.

Definition of the Definite Integral

The definition of the definite integral is based on the fact that for some functions, as the norm of the partitions of $[a, b]$ approaches zero, the values of the corresponding Riemann

sums approach a limiting value J . In particular, this is true for continuous and piecewise-continuous functions. We again use the symbol ε to represent a small positive number, and use it to specify how close to J the Riemann sum must be. The symbol δ is used for a second small positive number that specifies how small the norm of a partition must be in order for the Riemann sum to differ from J by no more than ε . We now define this limit precisely.

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^n f(c_k) \Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^n f(c_k) \Delta x_k - J \right| < \varepsilon.$$

The definition involves a limiting process in which the norm of the partition goes to zero. When this limit exists, the function f is said to be *integrable* over $[a, b]$.

We have many choices for a partition P with norm going to zero, and many choices of points c_k for each partition. The definite integral exists when we always get the same limit J , no matter what choices are made. When the limit exists we write

$$J = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k,$$

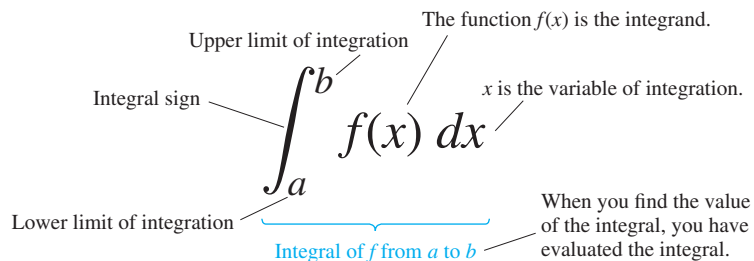
and we say that the *definite integral exists*.

Leibniz introduced a notation for the definite integral that captures its construction as a limit of Riemann sums. He envisioned the finite sums $\sum_{k=1}^n f(c_k) \Delta x_k$ becoming an infinite sum of function values $f(x)$ multiplied by “infinitesimal” subinterval widths dx . The sum symbol $\sum$ is replaced in the limit by the integral symbol $\int$, whose origin is in the letter “S” (for sum). The function values $f(c_k)$ are replaced by a continuous selection of function values $f(x)$. The subinterval widths Δx_k become the differential dx . It is as if we were summing all products of the form $f(x) \cdot dx$ as x goes from a to b . While this notation captures the underlying ideas, it is Riemann’s definition that gives a precise meaning to the definite integral.

If the definite integral exists, then instead of writing J , we write

$$\int_a^b f(x) dx.$$

We read this as “the integral from a to b of f of x dee x ” or sometimes as “the integral from a to b of f of x with respect to x .” The component parts in the integral symbol also have names:



When the definite integral exists, we say that the Riemann sums of f on $[a, b]$ **converge** to the definite integral $J = \int_a^b f(x) dx$ and that f is **integrable** over $[a, b]$.

If we choose all the subintervals in a partition to have equal width $\Delta x = (b - a)/n$, the Riemann sums have the form

$$S_n = \sum_{k=1}^n f(c_k) \Delta x_k = \sum_{k=1}^n f(c_k) \left(\frac{b-a}{n} \right), \quad \Delta x_k = \Delta x = (b-a)/n \text{ for all } k$$

where c_k is any point in the k th subinterval. If the definite integral exists, then these Riemann sums converge to the definite integral of f over $[a, b]$, so

$$J = \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \left(\frac{b-a}{n} \right). \quad \text{For equal-width subintervals, the condition } \|P\| \rightarrow 0 \text{ is equivalent to } n \rightarrow \infty.$$

If we pick the point c_k to be the right endpoint of the k th subinterval, so that $c_k = a + k \Delta x = a + k(b-a)/n$, then the formula for the definite integral becomes

The Definite Integral as a Limit of Riemann Sums with Equal-Width Subintervals

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f\left(a + k \frac{b-a}{n}\right) \left(\frac{b-a}{n}\right) \quad (1)$$

Equation (1) gives an explicit formula that can be used to compute definite integrals. When the definite integral exists, the Riemann sums coming from other choices of partitions and locations of points c_k will have the same limit as $n \rightarrow \infty$, provided that the norms of the partitions approach zero. Another choice of Riemann sums converging to the definite integral is given by the Midpoint Rule, discussed later in this section.

The value of the definite integral of a function over any particular interval depends on the function, not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x , we simply write the integral as

$$\int_a^b f(t) dt \quad \text{or} \quad \int_a^b f(u) du \quad \text{instead of} \quad \int_a^b f(x) dx.$$

No matter how we write the integral, it is still the same number, the limit of the Riemann sums as the norm of the partition approaches zero. Since it does not matter what letter we use, the variable of integration is called a **dummy variable**. In the three integrals given above, the dummy variables are t , u , and x .

Integrable and Nonintegrable Functions

Not every function defined over a closed interval $[a, b]$ is integrable even if the function is bounded. That is, the Riemann sums for some functions might not converge to the same limiting value, or to any value at all. Understanding which functions defined over $[a, b]$ are integrable and which are not requires advanced mathematical analysis, but fortunately most functions that commonly occur in applications are integrable. In particular, every continuous function over $[a, b]$ is integrable over this interval, and so is every function that has no more than a finite number of jump discontinuities on $[a, b]$. (See Figures 1.9 and 1.10. Such functions are called *piecewise continuous functions*, and they are defined in Additional Exercises 11–18 at the end of this chapter.) The following theorem, which is proved in more advanced courses, establishes these results.

THEOREM 1—Continuous Functions Are Integrable If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x) dx$ exists and f is integrable over $[a, b]$.

The idea behind Theorem 1 for continuous functions is given in Exercises 86 and 87. Briefly, when f is continuous, we can choose each c_k so that $f(c_k)$ gives the maximum value of f on the subinterval $[x_{k-1}, x_k]$, resulting in an upper sum. Likewise, we can choose c_k to give the minimum value of f on $[x_{k-1}, x_k]$ to obtain a lower sum. The upper and lower sums can be shown to converge to the same limiting value as the norm of the partition P tends to zero. Moreover, every Riemann sum is trapped between the values of the upper and lower sums, so every Riemann sum converges to the same limit as well. Therefore, the number J in the definition of the definite integral exists, and the continuous function f is integrable over $[a, b]$.

For integrability to fail, a function needs to be sufficiently discontinuous that the region between its graph and the x -axis cannot be approximated well by increasingly thin rectangles. Our first example is a function that is not integrable over a closed interval.

EXAMPLE 1 The function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

has no Riemann integral over $[0, 1]$. Underlying this is the fact that between any two numbers there are both a rational number and an irrational number. Thus the function jumps up and down too erratically over $[0, 1]$ to allow the region beneath its graph and above the x -axis to be approximated by rectangles, no matter how thin they are. In fact, we will show that upper sum approximations and lower sum approximations converge to different limiting values.

If we choose a partition P of $[0, 1]$, then the lengths of the intervals in the partition sum to 1; that is, $\sum_{k=1}^n \Delta x_k = 1$. In each subinterval $[x_{k-1}, x_k]$ there is a rational point, say c_k . Because c_k is rational, $f(c_k) = 1$. Since 1 is the maximum value that f can take anywhere, the upper sum approximation for this choice of c_k 's is

$$U = \sum_{k=1}^n f(c_k) \Delta x_k = \sum_{k=1}^n (1) \Delta x_k = 1.$$

As the norm of the partition approaches 0, these upper sum approximations converge to 1 (because each approximation is equal to 1).

On the other hand, we could pick the c_k 's differently and get a different result. Each subinterval $[x_{k-1}, x_k]$ also contains an irrational point c_k , and for this choice $f(c_k) = 0$. Since 0 is the minimum value that f can take anywhere, this choice of c_k gives us the minimum value of f on the subinterval. The corresponding lower sum approximation is

$$L = \sum_{k=1}^n f(c_k) \Delta x_k = \sum_{k=1}^n (0) \Delta x_k = 0.$$

These lower sum approximations converge to 0 as the norm of the partition converges to 0 (because they each equal 0).

Thus making different choices for the points c_k results in different limits for the corresponding Riemann sums. We conclude that the definite integral of f over the interval $[0, 1]$ does not exist and that f is not integrable over $[0, 1]$. ■

Theorem 1 says nothing about how to *calculate* definite integrals. A method of calculation will be developed in Section 5.4, through a connection of definite integrals to antiderivatives. Meanwhile, finite approximations can be used to calculate an approximation for a definite integral.

The Midpoint Rule

When setting up a Riemann sum

$$\sum_{k=1}^n f(c_k) \Delta x_k$$

we have many options on how to choose a partition and where to choose a point c_k in the k th interval of the partition. The simplest choice for a partition with n subintervals is to take each subinterval to have equal length $\Delta x = (b - a)/n$. In Equation (1) we obtained a formula by choosing c_k at the right endpoint of the k th interval. Setting c_k to be at the middle of the k th interval is often a better choice for approximating the integral using Riemann sums, as we saw in Figure 5.3. The midpoint is a good choice because on a small interval, the graph of a differentiable function can be approximated by a linear function, and the value of a linear function at an interval's midpoint is its average value over that interval.

With these choices, $[x_{k-1}, x_k] = [a + (k - 1)\Delta x, a + k\Delta x]$ is the k th interval in the partition, its midpoint is located at $c_k = (x_{k-1} + x_k)/2$, and its width is $\Delta x = (b - a)/n$. The midpoint rule for forming Riemann sums to approximate a definite integral takes the following form.

Midpoint Rule for Approximating a Definite Integral

$$\int_a^b f(x) dx \approx \sum_{k=1}^n f(c_k) \left(\frac{b-a}{n} \right) = [f(c_1) + f(c_2) + \cdots + f(c_n)] \left(\frac{b-a}{n} \right)$$

with $c_k = \frac{x_{k-1} + x_k}{2}$ and $x_k = a + k \left(\frac{b-a}{n} \right)$.

EXAMPLE 2 Approximate $\int_0^1 \frac{1}{1+x^2} dx$ using the midpoint rule with five intervals of equal length.

Solution The five intervals of the partition are $[0, 0.2]$, $[0.2, 0.4]$, $[0.4, 0.6]$, $[0.6, 0.8]$, and $[0.8, 1.0]$, each of length $1/5$. Their midpoints are at 0.1, 0.3, 0.5, 0.7, and 0.9. Applying the midpoint rule gives

$$\begin{aligned} \int_0^1 \frac{1}{1+x^2} dx &\approx [f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \left(\frac{1}{5} \right) \\ &= \left[\frac{1}{1.01} + \frac{1}{1.09} + \frac{1}{1.25} + \frac{1}{1.49} + \frac{1}{1.81} \right] (0.2) \\ &\approx 0.786. \end{aligned}$$

The midpoint rule becomes more powerful when we can combine it with an error bound that tells us how close the approximation it gives with n intervals is to the integral. In Chapter 8 we will examine error bounds in approximations of integrals.

Properties of Definite Integrals

In defining $\int_a^b f(x) dx$ as a limit of sums $\sum_{k=1}^n f(c_k) \Delta x_k$, we moved from left to right across the interval $[a, b]$. What would happen if we instead move right to left, starting with $x_0 = b$ and ending at $x_n = a$? Each Δx_k in the Riemann sum would change its sign, with $x_k - x_{k-1}$ now negative instead of positive. With the same choices of c_k in each subinterval, the sign of any Riemann sum would change, as would the sign of the limit, the integral $\int_b^a f(x) dx$. Since we have not previously given a meaning to integrating backward, we are led to define

$$\int_b^a f(x) dx = -\int_a^b f(x) dx. \quad a \text{ and } b \text{ interchanged}$$

It is convenient to have a definition for the integral over $[a, b]$ when $a = b$, so that we are computing an integral over an interval of zero width. Since $a = b$ gives $\Delta x = 0$, whenever $f(a)$ exists we define

$$\int_a^a f(x) dx = 0. \quad a \text{ is both the lower and the upper limit of integration.}$$

TABLE 5.6 Rules satisfied by definite integrals

1. <i>Order of Integration:</i>	$\int_b^a f(x) dx = -\int_a^b f(x) dx$	A definition
2. <i>Zero Width Interval:</i>	$\int_a^a f(x) dx = 0$	A definition when $f(a)$ exists
3. <i>Constant Multiple:</i>	$\int_a^b k f(x) dx = k \int_a^b f(x) dx$	Any constant k
4. <i>Sum and Difference:</i>	$\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$	
5. <i>Additivity:</i>	$\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$	$[a, c] \cup [c, b] = [a, b]$
6. <i>Max-Min Inequality:</i>	If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then $(\min f) \cdot (b - a) \leq \int_a^b f(x) dx \leq (\max f) \cdot (b - a).$	
7. <i>Domination:</i>	If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$. If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x) dx \geq 0$. Special case of Rule 6.	

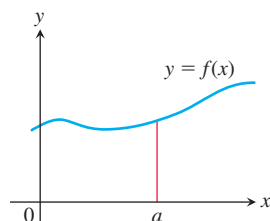
Theorem 2 states some basic properties of integrals, including the two just discussed. These properties, listed in Table 5.6, are very useful for computing integrals. We will refer to them repeatedly to simplify our calculations. Rules 2 through 7 have geometric interpretations, which are shown in Figure 5.11. The graphs in these figures show only positive functions, but the rules apply to general integrable functions, which could take both positive and negative values.

THEOREM 2 When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules listed in Table 5.6.

Rules 1 and 2 are definitions, but Rules 3 to 7 of Table 5.6 must be proved. Below we give a proof of Rule 6. Similar proofs can be given to verify the other properties in Table 5.6.

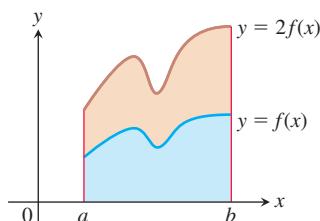
Proof of Rule 6 Rule 6 says that the integral of f over $[a, b]$ is never smaller than the minimum value of f times the length of the interval and never larger than the maximum value of f times the length of the interval. The reason is that for every partition of $[a, b]$ and for every choice of the points c_k ,

$$\begin{aligned}
 (\min f) \cdot (b - a) &= (\min f) \cdot \sum_{k=1}^n \Delta x_k & \sum_{k=1}^n \Delta x_k &= b - a \\
 &= \sum_{k=1}^n (\min f) \cdot \Delta x_k & & \text{Constant Multiple Rule} \\
 &\leq \sum_{k=1}^n f(c_k) \Delta x_k & & \min f \leq f(c_k) \\
 &\leq \sum_{k=1}^n (\max f) \cdot \Delta x_k & & f(c_k) \leq \max f \\
 &= (\max f) \cdot \sum_{k=1}^n \Delta x_k & & \text{Constant Multiple Rule} \\
 &= (\max f) \cdot (b - a).
 \end{aligned}$$

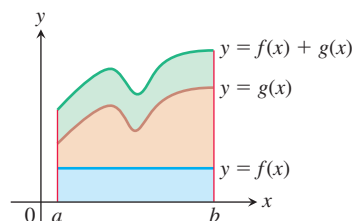


(a) Zero Width Interval:

$$\int_a^a f(x) dx = 0$$

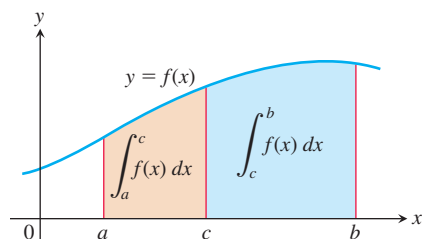
(b) Constant Multiple: ($k = 2$)

$$\int_a^b k f(x) dx = k \int_a^b f(x) dx$$



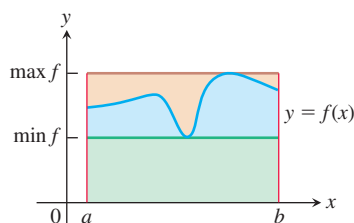
(c) Sum: (areas add)

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$



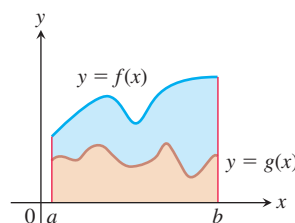
(d) Additivity for Definite Integrals:

$$\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$$



(e) Max-Min Inequality:

$$\begin{aligned} (\min f) \cdot (b - a) &\leq \int_a^b f(x) dx \\ &\leq (\max f) \cdot (b - a) \end{aligned}$$



(f) Domination:

$$\begin{aligned} \text{If } f(x) \geq g(x) \text{ on } [a, b], \text{ then} \\ \int_a^b f(x) dx \geq \int_a^b g(x) dx. \end{aligned}$$

FIGURE 5.11 Geometric interpretations of Rules 2–7 in Table 5.6.

In short, all Riemann sums for f on $[a, b]$ satisfy the inequalities

$$(\min f) \cdot (b - a) \leq \sum_{k=1}^n f(c_k) \Delta x_k \leq (\max f) \cdot (b - a).$$

Hence their limit, which is the integral, satisfies the same inequalities. ■

EXAMPLE 3 To illustrate some of the rules, we suppose that

$$\int_{-1}^1 f(x) dx = 5, \quad \int_1^4 f(x) dx = -2, \quad \text{and} \quad \int_{-1}^1 h(x) dx = 7.$$

Then

$$1. \int_4^1 f(x) dx = -\int_1^4 f(x) dx = -(-2) = 2 \quad \text{Rule 1}$$

$$\begin{aligned} 2. \int_{-1}^1 [2f(x) + 3h(x)] dx &= 2 \int_{-1}^1 f(x) dx + 3 \int_{-1}^1 h(x) dx \\ &= 2(5) + 3(7) = 31 \end{aligned} \quad \text{Rules 3 and 4}$$

$$3. \int_{-1}^4 f(x) dx = \int_{-1}^1 f(x) dx + \int_1^4 f(x) dx = 5 + (-2) = 3 \quad \text{Rule 5} \quad \blacksquare$$

EXAMPLE 4 Show that the value of $\int_0^1 \sqrt{1 + \cos x} dx$ is less than or equal to $\sqrt{2}$.

Solution The Max-Min Inequality for definite integrals (Rule 6) says that $(\min f) \cdot (b - a)$ is a *lower bound* for the value of $\int_a^b f(x) dx$ and that $(\max f) \cdot (b - a)$ is an *upper bound*. The maximum value of $\sqrt{1 + \cos x}$ on $[0, 1]$ is $\sqrt{1 + 1} = \sqrt{2}$, so

$$\int_0^1 \sqrt{1 + \cos x} dx \leq \sqrt{2} \cdot (1 - 0) = \sqrt{2}. \quad \blacksquare$$

Area Under the Graph of a Nonnegative Function

We now return to the problem that started this chapter, which is defining what we mean by the *area* of a region having a curved boundary. In Section 5.1 we approximated the area under the graph of a nonnegative continuous function using several types of finite sums of areas of rectangles that approximate the region—upper sums, lower sums, and sums using the midpoints of each subinterval—all of which are Riemann sums constructed in special ways. Theorem 1 guarantees that all of these Riemann sums converge to a single definite integral as the norm of the partitions approaches zero and the number of subintervals goes to infinity. As a result, we can now *define* the area under the graph of a nonnegative integrable function to be the value of that definite integral.

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b ,

$$A = \int_a^b f(x) \, dx.$$

For the first time, we have a rigorous definition for the area of a region whose boundary is the graph of a continuous function. We now apply this to a simple example, the area under a straight line, and we verify that our new definition agrees with our previous notion of area.

EXAMPLE 5 Compute $\int_0^b x \, dx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.

Solution The region of interest is a triangle (Figure 5.12). We compute the area in two ways.

- (a) To compute the definite integral as the limit of Riemann sums, we calculate $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k$ for partitions whose norms go to zero. Theorem 1 tells us that it does not matter how we choose the partitions or the points c_k as long as the norms approach zero. All choices give the exact same limit. So we consider the partition P that subdivides the interval $[0, b]$ into n subintervals of equal width $\Delta x = (b - 0)/n = b/n$, and we choose c_k to be the right endpoint in each subinterval as in formula (1). The partition is $P = \left\{0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \dots, \frac{nb}{n}\right\}$ and $c_k = \frac{kb}{n}$. So

$$\begin{aligned} \sum_{k=1}^n f(c_k) \Delta x &= \sum_{k=1}^n \frac{kb}{n} \cdot \frac{b}{n} && f(c_k) = c_k \\ &= \sum_{k=1}^n \frac{kb^2}{n^2} \\ &= \frac{b^2}{n^2} \sum_{k=1}^n k && \text{Constant Multiple Rule} \\ &= \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2} && \text{Sum of first } n \text{ integers} \\ &= \frac{b^2}{2} \left(1 + \frac{1}{n}\right). \end{aligned}$$

As $n \rightarrow \infty$ and $\|P\| \rightarrow 0$, this last expression on the right has the limit $b^2/2$. Therefore,

$$\int_0^b x \, dx = \frac{b^2}{2}.$$

- (b) Since the area equals the definite integral for a nonnegative function, we can quickly derive the definite integral by using the formula for the area of a triangle having base

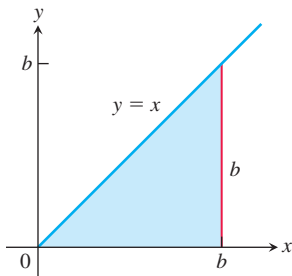


FIGURE 5.12 The region in Example 5 is a triangle.

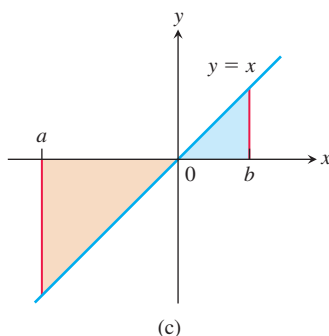
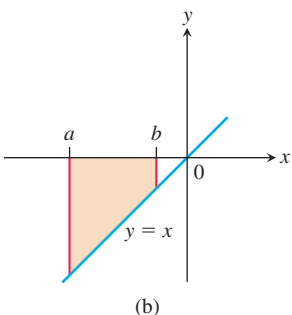
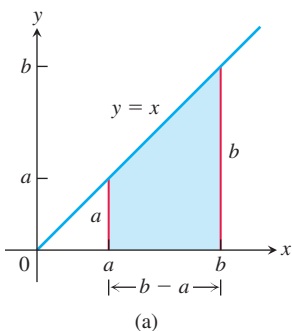


FIGURE 5.13 (a) The area of this trapezoidal region is $A = (b^2 - a^2)/2$. (b) The definite integral in Equation (2) gives the negative of the area of this trapezoidal region. (c) The definite integral in Equation (2) gives the area of the blue triangular region added to the negative of the area of the tan triangular region.

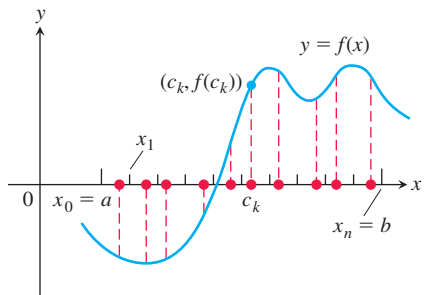


FIGURE 5.14 A sample of values of a function on an interval $[a, b]$.

length b and height $y = b$. The area is $A = (1/2)b \cdot b = b^2/2$. Again we conclude that $\int_0^b x \, dx = b^2/2$. ■

Example 5 can be generalized to integrate $f(x) = x$ over any closed interval $[a, b]$ for which $0 < a < b$.

First write

$$\int_0^b x \, dx = \int_0^a x \, dx + \int_a^b x \, dx \quad \text{Rule 5}$$

Then, by rearranging this equation and applying Example 5, we obtain

$$\int_a^b x \, dx = \int_0^b x \, dx - \int_0^a x \, dx = \frac{b^2}{2} - \frac{a^2}{2}. \quad \text{Example 5}$$

In conclusion, we have the following rule for integrating $f(x) = x$ over the interval $[a, b]$:

$$\int_a^b x \, dx = \frac{b^2}{2} - \frac{a^2}{2}, \quad a < b \quad (2)$$

This computation gives the area of the trapezoid in Figure 5.13a. Equation (2) remains valid when a and b are negative, but the interpretation of the definite integral changes. When $a < b < 0$, the definite integral value $(b^2 - a^2)/2$ is a negative number, the negative of the area of a trapezoid dropping down to the line $y = x$ below the x -axis (Figure 5.13b). When $a < 0$ and $b > 0$, Equation (2) is still valid and the definite integral gives the difference between two areas, the area under the graph and above $[0, b]$ minus the area below $[a, 0]$ and over the graph (Figure 5.13c).

The following results can also be established by using a Riemann sum calculation similar to the one we used in Example 5 (Exercises 63 and 65).

$$\int_a^b c \, dx = c(b - a), \quad c \text{ any constant} \quad (3)$$

$$\int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3}, \quad a < b \quad (4)$$

Average Value of a Continuous Function Revisited

In Section 5.1 we informally introduced the average value of a nonnegative continuous function f over an interval $[a, b]$, leading us to define this average as the area under the graph of $y = f(x)$ divided by $b - a$. In integral notation we write this as

$$\text{Average} = \frac{1}{b - a} \int_a^b f(x) \, dx.$$

This formula gives us a precise definition of the average value of a continuous (or integrable) function, whether it is positive, negative, or both.

Alternatively, we can justify this formula through the following reasoning. We start with the idea from arithmetic that the average of n numbers is their sum divided by n . A continuous function f on $[a, b]$ may have infinitely many values, but we can still sample them in an orderly way. We divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$ and evaluate f at a point c_k in each (Figure 5.14). The average of the n sampled values is

$$\begin{aligned} \frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} &= \frac{1}{n} \sum_{k=1}^n f(c_k) \\ &= \frac{\Delta x}{b - a} \sum_{k=1}^n f(c_k) & \Delta x = \frac{b - a}{n}, \text{ so } \frac{1}{n} = \frac{\Delta x}{b - a} \\ &= \frac{1}{b - a} \sum_{k=1}^n f(c_k) \Delta x. & \text{Constant Multiple Rule} \end{aligned}$$

The average of the samples is obtained by dividing a Riemann sum for f on $[a, b]$ by $(b - a)$. As we increase the number of samples and let the norm of the partition approach zero, the average approaches $(1/(b - a)) \int_a^b f(x) dx$. Both points of view lead us to the following definition.

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$** , also called its **mean**, is

$$\text{av}(f) = \frac{1}{b - a} \int_a^b f(x) dx.$$

EXAMPLE 6 Find the average value of $f(x) = x$ on $[1, 3]$.

Solution From Equation (2) we see that $\int_1^3 x dx = 9/2 - 1/2 = 4$.

So the average value of f on the interval $[1, 3]$ is $(1/2)(4) = 2$. ■

EXAMPLE 7 Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

Solution We recognize $f(x) = \sqrt{4 - x^2}$ as the function whose graph is the upper semicircle of radius 2 centered at the origin (Figure 5.15).

Since we know the area inside a circle, we do not need to take the limit of Riemann sums. The area between the semicircle and the x -axis from -2 to 2 can be computed using the geometry formula

$$\text{Area} = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi (2)^2 = 2\pi.$$

Because f is nonnegative, the area is also the value of the integral of f from -2 to 2 ,

$$\int_{-2}^2 \sqrt{4 - x^2} dx = 2\pi.$$

Therefore, the average value of f is

$$\text{av}(f) = \frac{1}{2 - (-2)} \int_{-2}^2 \sqrt{4 - x^2} dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$

Notice that the average value of f over $[-2, 2]$ is the same as the height of a rectangle over $[-2, 2]$ whose area equals the area of the upper semicircle (see Figure 5.15). ■

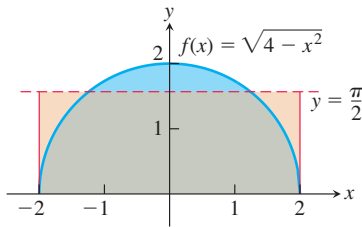


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 7). The area of the rectangle shown here is $4 \cdot (\pi/2) = 2\pi$, which is also the area of the semicircle.

EXERCISES 5.3

Interpreting Limits of Sums as Integrals

Express the limits in Exercises 1–8 as definite integrals.

- $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n c_k^2 \Delta x_k$, where P is a partition of $[0, 2]$
- $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n 2c_k^3 \Delta x_k$, where P is a partition of $[-1, 0]$
- $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (c_k^2 - 3c_k) \Delta x_k$, where P is a partition of $[-7, 5]$
- $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \left(\frac{1}{c_k} \right) \Delta x_k$, where P is a partition of $[1, 4]$
- $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \frac{1}{1 - c_k} \Delta x_k$, where P is a partition of $[2, 3]$

$$6. \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \sqrt{4 - c_k^2} \Delta x_k, \text{ where } P \text{ is a partition of } [0, 1]$$

$$7. \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (\sec c_k) \Delta x_k, \text{ where } P \text{ is a partition of } [-\pi/4, 0]$$

$$8. \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (\tan c_k) \Delta x_k, \text{ where } P \text{ is a partition of } [0, \pi/4]$$

Using the Definite Integral Rules

9. Suppose that f and g are integrable and that

$$\int_1^2 f(x) dx = -4, \quad \int_1^5 f(x) dx = 6, \quad \int_1^5 g(x) dx = 8.$$

Use the rules in Table 5.6 to find

- a. $\int_2^2 g(x) dx$ b. $\int_5^1 g(x) dx$
 c. $\int_1^2 3f(x) dx$ d. $\int_2^5 f(x) dx$
 e. $\int_1^5 [f(x) - g(x)] dx$ f. $\int_1^5 [4f(x) - g(x)] dx$

10. Suppose that f and h are integrable and that

$$\int_1^9 f(x) dx = -1, \quad \int_7^9 f(x) dx = 5, \quad \int_7^9 h(x) dx = 4.$$

Use the rules in Table 5.6 to find

- a. $\int_1^9 -2f(x) dx$ b. $\int_7^9 [f(x) + h(x)] dx$
 c. $\int_7^9 [2f(x) - 3h(x)] dx$ d. $\int_9^1 f(x) dx$
 e. $\int_1^7 f(x) dx$ f. $\int_9^7 [h(x) - f(x)] dx$

11. Suppose that $\int_1^2 f(x) dx = 5$. Find

- a. $\int_1^2 f(u) du$ b. $\int_1^2 \sqrt{3} f(z) dz$
 c. $\int_2^1 f(t) dt$ d. $\int_1^2 [-f(x)] dx$

12. Suppose that $\int_{-3}^0 g(t) dt = \sqrt{2}$. Find

- a. $\int_0^{-3} g(t) dt$ b. $\int_{-3}^0 g(u) du$
 c. $\int_{-3}^0 [-g(x)] dx$ d. $\int_{-3}^0 \frac{g(r)}{\sqrt{2}} dr$

13. Suppose that f is integrable and that $\int_0^3 f(z) dz = 3$ and $\int_0^4 f(z) dz = 7$. Find

- a. $\int_3^4 f(z) dz$ b. $\int_4^3 f(t) dt$

14. Suppose that h is integrable and that $\int_{-1}^1 h(r) dr = 0$ and $\int_{-1}^3 h(r) dr = 6$. Find

- a. $\int_1^3 h(r) dr$ b. $-\int_3^1 h(u) du$

Using Known Areas to Find Integrals

In Exercises 15–22, graph the integrands and use known area formulas to evaluate the integrals.

15. $\int_{-2}^4 \left(\frac{x}{2} + 3\right) dx$ 16. $\int_{1/2}^{3/2} (-2x + 4) dx$
 17. $\int_{-3}^3 \sqrt{9 - x^2} dx$ 18. $\int_{-4}^0 \sqrt{16 - x^2} dx$
 19. $\int_{-2}^1 |x| dx$ 20. $\int_{-1}^1 (1 - |x|) dx$
 21. $\int_{-1}^1 (2 - |x|) dx$ 22. $\int_{-1}^1 (1 + \sqrt{1 - x^2}) dx$

Use known area formulas to evaluate the integrals in Exercises 23–28.

23. $\int_0^b \frac{x}{2} dx, \quad b > 0$ 24. $\int_0^b 4x dx, \quad b > 0$
 25. $\int_a^b 2s ds, \quad 0 < a < b$ 26. $\int_a^b 3t dt, \quad 0 < a < b$

27. $f(x) = \sqrt{4 - x^2}$ on a. $[-2, 2]$, b. $[0, 2]$

28. $f(x) = 3x + \sqrt{1 - x^2}$ on a. $[-1, 0]$, b. $[-1, 1]$

Evaluating Definite Integrals

Use the results of Equations (2) and (4) to evaluate the integrals in Exercises 29–40.

29. $\int_1^{\sqrt{2}} x dx$ 30. $\int_{0.5}^{2.5} x dx$ 31. $\int_{\pi}^{2\pi} \theta d\theta$
 32. $\int_{\sqrt{2}}^{5\sqrt{2}} r dr$ 33. $\int_0^{\sqrt[3]{7}} x^2 dx$ 34. $\int_0^{0.3} s^2 ds$
 35. $\int_0^{1/2} t^2 dt$ 36. $\int_0^{\pi/2} \theta^2 d\theta$ 37. $\int_a^{2a} x dx$
 38. $\int_a^{\sqrt{3}} x dx$ 39. $\int_0^{\sqrt[3]{b}} x^2 dx$ 40. $\int_0^{3b} x^2 dx$

Use the rules in Table 5.6 and Equations (2)–(4) to evaluate the integrals in Exercises 41–50.

41. $\int_3^1 7 dx$ 42. $\int_0^2 5x dx$
 43. $\int_0^2 (2t - 3) dt$ 44. $\int_0^{\sqrt{2}} (t - \sqrt{2}) dt$
 45. $\int_2^1 \left(1 + \frac{z}{2}\right) dz$ 46. $\int_3^0 (2z - 3) dz$
 47. $\int_1^2 3u^2 du$ 48. $\int_{1/2}^1 24u^2 du$
 49. $\int_0^2 (3x^2 + x - 5) dx$ 50. $\int_1^2 (3x^2 + x - 5) dx$

Finding Area by Definite Integrals

In Exercises 51–54, use a definite integral to find the area of the region between the given curve and the x -axis on the interval $[0, b]$.

51. $y = 3x^2$ 52. $y = \pi x^2$
 53. $y = 2x$ 54. $y = \frac{x}{2} + 1$

Finding Average Value

In Exercises 55–62, graph the function and find its average value over the given interval.

55. $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$ 56. $f(x) = -\frac{x^2}{2}$ on $[0, 3]$
 57. $f(x) = -3x^2 - 1$ on $[0, 1]$ 58. $f(x) = 3x^2 - 3$ on $[0, 1]$
 59. $f(t) = (t - 1)^2$ on $[0, 3]$ 60. $f(t) = t^2 - t$ on $[-2, 1]$
 61. $g(x) = |x| - 1$ on a. $[-1, 1]$, b. $[1, 3]$, and c. $[-1, 3]$
 62. $h(x) = -|x|$ on a. $[-1, 0]$, b. $[0, 1]$, and c. $[-1, 1]$

Definite Integrals as Limits of Sums

Use the method of Example 5a or Equation (1) to evaluate the definite integrals in Exercises 63–70.

63. $\int_a^b c dx$ 64. $\int_0^2 (2x + 1) dx$
 65. $\int_a^b x^2 dx, \quad a < b$ 66. $\int_{-1}^0 (x - x^2) dx$
 67. $\int_{-1}^2 (3x^2 - 2x + 1) dx$ 68. $\int_{-1}^1 x^3 dx$
 69. $\int_a^b x^3 dx, \quad a < b$ 70. $\int_0^1 (3x - x^3) dx$

Theory and Examples

71. What values of a and b , with $a < b$, maximize the value of

$$\int_a^b (x - x^2) dx?$$

(Hint: Where is the integrand positive?)

72. What values of a and b , with $a < b$, minimize the value of

$$\int_a^b (x^4 - 2x^2) dx?$$

73. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1+x^2} dx.$$

74. (Continuation of Exercise 73.) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1+x^2} dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1+x^2} dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1+x^2} dx.$$

75. Show that the value of $\int_0^1 \sin(x^2) dx$ cannot possibly be 2.
 76. Show that the value of $\int_0^1 \sqrt{x+8} dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.
 77. **Integrals of nonnegative functions** Use the Max-Min Inequality to show that if f is integrable, then

$$f(x) \geq 0 \quad \text{on} \quad [a, b] \Rightarrow \int_a^b f(x) dx \geq 0.$$

78. **Integrals of nonpositive functions** Show that if f is integrable, then

$$f(x) \leq 0 \quad \text{on} \quad [a, b] \Rightarrow \int_a^b f(x) dx \leq 0.$$

79. Use the inequality $\sin x \leq x$, which holds for $x \geq 0$, to find an upper bound for the value of $\int_0^1 \sin x dx$.
 80. The inequality $\sec x \geq 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x dx$.
 81. If $\text{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the constant function $\text{av}(f)$ should have the same integral over $[a, b]$ as f . Does it? That is, does

$$\int_a^b \text{av}(f) dx = \int_a^b f(x) dx?$$

Give reasons for your answer.

82. It would be nice if average values of integrable functions obeyed the following rules on an interval $[a, b]$.

- a. $\text{av}(f + g) = \text{av}(f) + \text{av}(g)$
- b. $\text{av}(kf) = k \text{av}(f)$ (any number k)
- c. $\text{av}(f) \leq \text{av}(g)$ if $f(x) \leq g(x)$ on $[a, b]$.

Do these rules ever hold? Give reasons for your answers.

83. Upper and lower sums for increasing functions

- a. Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$. Show by referring to the accompanying

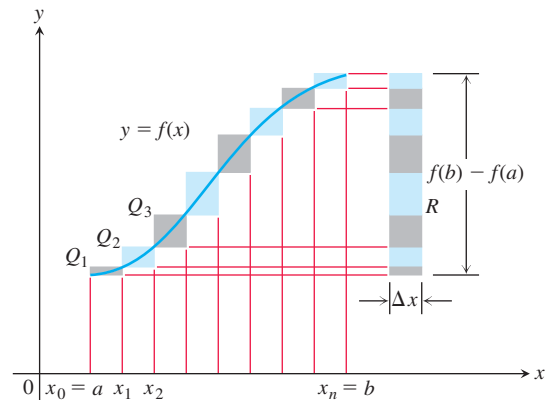
figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are $[f(b) - f(a)]$ by Δx . (Hint: The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \dots, Q_{n-1}Q_n$ lie approximately along the curve. There is no overlapping when these rectangles are shifted horizontally onto R .)

- b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of the partition of $[a, b]$ vary in size. Show that

$$U - L \leq |f(b) - f(a)| \Delta x_{\max},$$

where $\Delta x_{\max}$ is the norm of P , and that hence

$$\lim_{\|P\| \rightarrow 0} (U - L) = 0.$$



84. Upper and lower sums for decreasing functions (Continuation of Exercise 83.)

- a. Draw a figure like the one in Exercise 83 for a continuous function $f(x)$ whose values decrease steadily as x moves from left to right across the interval $[a, b]$. Let P be a partition of $[a, b]$ into subintervals of equal length. Find an expression for $U - L$ that is analogous to the one you found for $U - L$ in Exercise 83a.

- b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of P vary in size. Show that the inequality

$$U - L \leq |f(b) - f(a)| \Delta x_{\max}$$

of Exercise 83b still holds and hence $\lim_{\|P\| \rightarrow 0} (U - L) = 0$.

85. Use the formula

$$\begin{aligned} \sin h + \sin 2h + \sin 3h + \cdots + \sin kh \\ = \frac{\cos(h/2) - \cos((k + (1/2))h)}{2 \sin(h/2)} \end{aligned}$$

to find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$ in two steps:

- a. Partition the interval $[0, \pi/2]$ into n subintervals of equal length and calculate the corresponding upper sum U ; then
- b. Find the limit of U as $n \rightarrow \infty$ and $\Delta x = (b - a)/n \rightarrow 0$.

86. Suppose that f is continuous and nonnegative over $[a, b]$, as in the accompanying figure. By inserting points

$$x_1, x_2, \dots, x_{k-1}, x_k, \dots, x_{n-1}$$

as shown, divide $[a, b]$ into n subintervals of lengths $\Delta x_1 = x_1 - a, \Delta x_2 = x_2 - x_1, \dots, \Delta x_n = b - x_{n-1}$, which need not be equal.

- a. If $m_k = \min \{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **lower sum**

$$L = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n$$

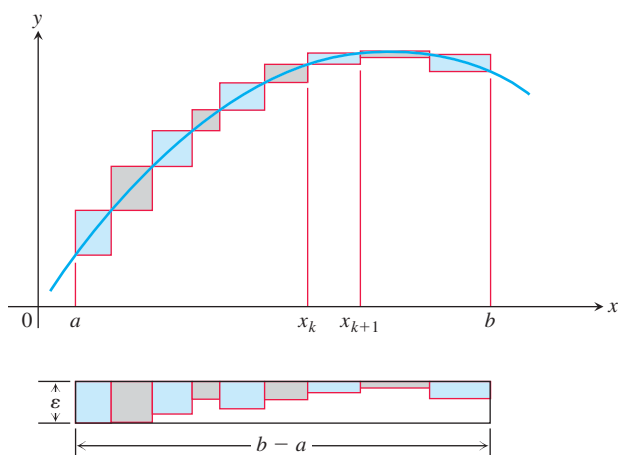
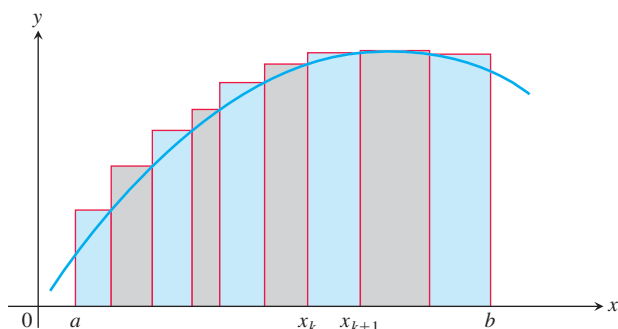
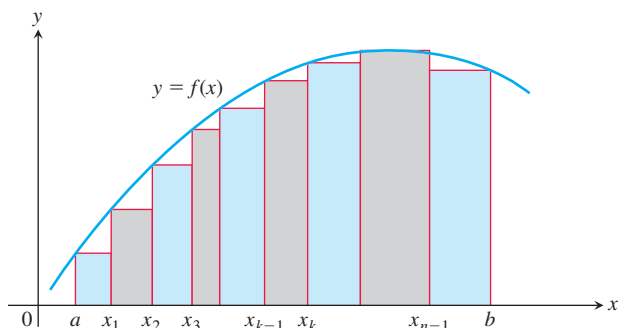
and the shaded regions in the first part of the figure.

- b. If $M_k = \max \{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **upper sum**

$$U = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n$$

and the shaded regions in the second part of the figure.

- c. Explain the connection between $U - L$ and the shaded regions along the curve in the third part of the figure.



87. We say f is **uniformly continuous** on $[a, b]$ if, given any $\varepsilon > 0$, there is a $\delta > 0$ such that if x_1, x_2 are in $[a, b]$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \varepsilon$. It can be shown that a continuous function on $[a, b]$ is uniformly continuous. Use this and the figure for Exercise 86 to show that if f is continuous and $\varepsilon > 0$ is given, it is possible to make $U - L \leq \varepsilon \cdot (b - a)$ by making the largest of the Δx_k 's sufficiently small.

88. If you average 48 km/h on a 240-km trip and then return over the same 240 km at the rate of 80 km/h, what is your average speed for the trip? Give reasons for your answer.

89. Integrals of functions that are equal except at one point

Suppose that $f(x)$ is a continuous function over the interval $[a, b]$ and that $g(x)$ is a function on $[a, b]$ such that $g(x) = f(x)$ except at a single point $c \in [a, b]$. Show that $g(x)$ is also integrable over $[a, b]$ and that $\int_a^b g(x) dx = \int_a^b f(x) dx$. (Hint: For a given n , by how much can two different Riemann sums as given in Equation (1) differ?)

90. Some integrable functions that are not continuous

- a. The floor function $f(x) = \lfloor x \rfloor$ gives the greatest integer smaller than or equal to x (Example 5 of Section 1.1). This function is not continuous on the interval $[1, 3]$. Show that f is integrable on $[1, 3]$ and that $\int_1^3 \lfloor x \rfloor dx = 3$.
- b. The function $f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 2 & \text{if } 0 \leq x, \end{cases}$ is not continuous on the interval $[-1, 1]$. Show that f is integrable on $[-1, 1]$ and that $\int_{-1}^1 f(x) dx = 1$.

COMPUTER EXPLORATIONS

If your CAS can draw rectangles associated with Riemann sums, use it to draw rectangles associated with Riemann sums that converge to the integrals in Exercises 91–96. Use $n = 4, 10, 20$, and 50 subintervals of equal length in each case.

91. $\int_0^1 (1 - x) dx = \frac{1}{2}$ 92. $\int_0^1 (x^2 + 1) dx = \frac{4}{3}$
93. $\int_{-\pi}^{\pi} \cos x dx = 0$ 94. $\int_0^{\pi/4} \sec^2 x dx = 1$
95. $\int_{-1}^1 |x| dx = 1$
96. $\int_1^2 \frac{1}{x} dx$ (The integral's value is about 0.693.)

In Exercises 97–104, use a CAS to perform the following steps:

- a. Plot the functions over the given interval.
- b. Partition the interval into $n = 100, 200$, and 1000 subintervals of equal length, and evaluate the function at the midpoint of each subinterval.
- c. Compute the average value of the function values generated in part (b).
- d. Solve the equation $f(x) = (\text{average value})$ for x using the average value calculated in part (c) for the $n = 1000$ partitioning.
97. $f(x) = \sin x$ on $[0, \pi]$
98. $f(x) = \sin^2 x$ on $[0, \pi]$
99. $f(x) = x \sin \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$
100. $f(x) = x \sin^2 \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$
101. $f(x) = xe^{-x}$ on $[0, 1]$
102. $f(x) = e^{-x^2}$ on $[0, 1]$
103. $f(x) = \frac{\ln x}{x}$ on $[2, 5]$
104. $f(x) = \frac{1}{\sqrt{1 - x^2}}$ on $\left[0, \frac{1}{2}\right]$

5.4 The Fundamental Theorem of Calculus

HISTORICAL BIOGRAPHY

Sir Isaac Newton
(1642–1727)

In his youth in England, Newton was interested in mechanical devices and their underlying theories. He even constructed lanterns and windmills that he designed. During the 1670s and 1680s, he built his reputation as a scientific genius. His contributions included the theory of universal gravitation, the laws of motion, methods of calculus, and the composition of white light.

To know more, visit the companion Website.

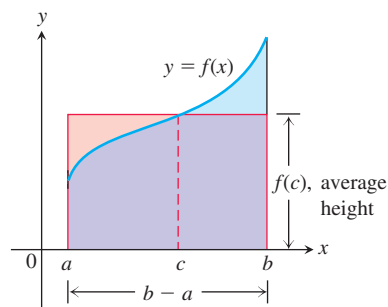


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b ,

$$f(c)(b - a) = \int_a^b f(x) dx.$$

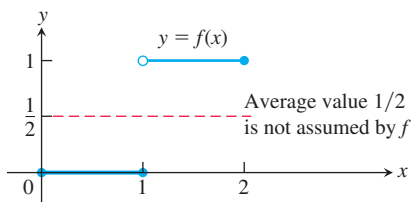


FIGURE 5.17 A discontinuous function need not assume its average value.

In this section we present the Fundamental Theorem of Calculus, which is the central theorem of integral calculus. It connects integration and differentiation, enabling us to compute integrals by using an antiderivative of the integrand function, rather than by taking limits of Riemann sums as we did in Section 5.3. Leibniz and Newton exploited this relationship and started mathematical developments that fueled the scientific revolution for the next 200 years.

Along the way, we will present an integral version of the Mean Value Theorem, which is another important theorem of integral calculus and is used to prove the Fundamental Theorem. We also find that the net change of a function over an interval is the integral of its rate of change, as suggested by Example 2 in Section 5.1.

Mean Value Theorem for Definite Integrals

In the previous section we defined the average value of a continuous function over a closed interval $[a, b]$ to be the definite integral $\int_a^b f(x) dx$ divided by the length or width $b - a$ of the interval. The Mean Value Theorem for Definite Integrals asserts that this average value is *always* taken on at least once by the function f in the interval.

The graph in Figure 5.16 shows a *positive* continuous function $y = f(x)$ defined over the interval $[a, b]$. Geometrically, the Mean Value Theorem says that there is a number c in $[a, b]$ such that the rectangle with height equal to the average value $f(c)$ of the function and base width $b - a$ has exactly the same area as the region beneath the graph of f from a to b .

THEOREM 3—The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b - a} \int_a^b f(x) dx.$$

Proof If we divide all three expressions in the Max-Min Inequality (Table 5.6, Rule 6) by $(b - a)$, we obtain

$$\min f \leq \frac{1}{b - a} \int_a^b f(x) dx \leq \max f.$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions (Section 2.6) says that f must assume every value between $\min f$ and $\max f$. It must therefore assume the value $(1/(b - a)) \int_a^b f(x) dx$ at some point c in $[a, b]$. ■

The continuity of f is important here. It is possible for a discontinuous function to never equal its average value (Figure 5.17).

EXAMPLE 1 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x) dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution The average value of f on $[a, b]$ is

$$\text{av}(f) = \frac{1}{b - a} \int_a^b f(x) dx = \frac{1}{b - a} \cdot 0 = 0.$$

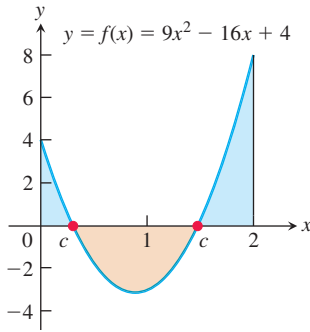


FIGURE 5.18 The function $f(x) = 9x^2 - 16x + 4$ satisfies $\int_0^2 f(x) dx = 0$, and there are two values of c in the interval $[0, 2]$ where $f(c) = 0$.

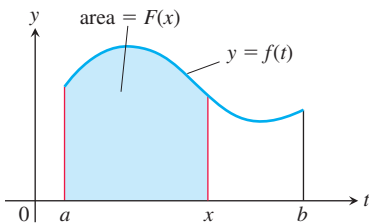


FIGURE 5.19 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

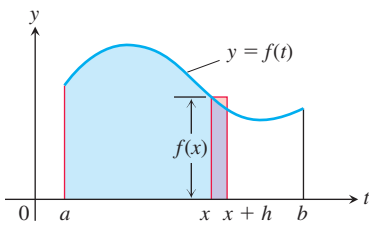


FIGURE 5.20 In Equation (1), $F(x)$ is the area to the left of x . Also, $F(x + h)$ is the area to the left of $x + h$. The difference quotient $[F(x + h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

By the Mean Value Theorem for Definite Integrals, f assumes this value at some point $c \in [a, b]$. This is illustrated in Figure 5.18 for the function $f(x) = 9x^2 - 16x + 4$ on the interval $[0, 2]$. ■

Fundamental Theorem, Part 1

It can be very difficult to compute definite integrals by taking the limit of Riemann sums. We now develop a powerful new method for evaluating definite integrals, based on using antiderivatives. This method combines the two strands of calculus. One strand involves the idea of taking the limits of finite sums to obtain a definite integral, and the other strand contains derivatives and antiderivatives. They come together in the Fundamental Theorem of Calculus. We begin by considering how to differentiate a certain type of function that is described as an integral.

If $f(t)$ is an integrable function over a finite interval I , then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is

$$F(x) = \int_a^x f(t) dt. \quad (1)$$

For example, if f is nonnegative and x lies to the right of a , then $F(x)$ is the area under the graph from a to x (Figure 5.19). The variable x is the upper limit of integration of an integral, but F is just like any other real-valued function of a real variable. For each value of the input x , there is a single numerical output, in this case the definite integral of f from a to x .

Equation (1) gives a useful way to define new functions (as we will see in Section 7.1), but its key importance is the connection that it makes between integrals and derivatives. If f is a continuous function, then the Fundamental Theorem asserts that F is a differentiable function of x whose derivative is f itself. That is, at each x in the interval $[a, b]$ we have

$$F'(x) = f(x).$$

To gain some insight into why this holds, we look at the geometry behind it.

If $f \geq 0$ on $[a, b]$, then to compute $F'(x)$ from the definition of the derivative, we must take the limit as $h \rightarrow 0$ of the difference quotient

$$\frac{F(x + h) - F(x)}{h}.$$

If $h > 0$, then $F(x + h)$ is the area under the graph of f from a to $x + h$, while $F(x)$ is the area under the graph of f from a to x . Subtracting the two gives us the area under the graph of f between x and $x + h$ (see Figure 5.20). As shown in Figure 5.20, if h is small, the area under the graph of f from x to $x + h$ is approximated by the area of the rectangle whose height is $f(x)$ and whose base is the interval $[x, x + h]$. That is,

$$F(x + h) - F(x) \approx hf(x).$$

Dividing both sides by h , we see that the value of the difference quotient is very close to the value of $f(x)$:

$$\frac{F(x + h) - F(x)}{h} \approx f(x).$$

This approximation improves as h approaches 0. It is reasonable to expect that $F'(x)$, which is the limit of this difference quotient as $h \rightarrow 0$, equals $f(x)$, so that

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x + h) - F(x)}{h} = f(x).$$

This equation is true even if the function f is not positive, and it forms the first part of the Fundamental Theorem of Calculus.

THEOREM 4—The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t) dt$ is continuous on $[a, b]$ and differentiable on (a, b) , and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x). \quad (2)$$

Before proving Theorem 4, we look at several examples to gain an understanding of what it says. In each of these examples, notice that the independent variable x appears in either the upper or the lower limit of integration (either as part of a formula or by itself). The independent variable on which y depends in these examples is x , while t is merely a dummy variable in the integral.

EXAMPLE 2 Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1) dt$

(b) $y = \int_x^5 3t \sin t dt$

(c) $y = \int_1^{x^2} \cos t dt$

(d) $y = \int_{1+3x^2}^4 \frac{1}{2 + e^t} dt$

Solution We calculate the derivatives with respect to the independent variable x .

(a) $\frac{dy}{dx} = \frac{d}{dx} \int_a^x (t^3 + 1) dt = x^3 + 1$ Eq. (2) with $f(t) = t^3 + 1$

(b) $\frac{dy}{dx} = \frac{d}{dx} \int_x^5 3t \sin t dt = \frac{d}{dx} \left(- \int_5^x 3t \sin t dt \right)$ Table 5.6, Rule 1
 $= - \frac{d}{dx} \int_5^x 3t \sin t dt$
 $= -3x \sin x$ Eq. (2) with $f(t) = 3t \sin t$

(c) The upper limit of integration is not x but x^2 . This makes y a composition of the two functions

$$y = \int_1^u \cos t dt \quad \text{and} \quad u = x^2.$$

We must therefore apply the Chain Rule to find dy/dx :

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \left(\frac{d}{du} \int_1^u \cos t dt \right) \cdot \frac{du}{dx} \\ &= \cos u \cdot \frac{du}{dx} && \text{Eq. (2) with } f(t) = \cos t \\ &= (\cos x^2) \cdot 2x \\ &= 2x \cos x^2 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \frac{d}{dx} \int_{1+3x^2}^4 \frac{1}{2+e^t} dt &= \frac{d}{dx} \left(- \int_4^{1+3x^2} \frac{1}{2+e^t} dt \right) && \text{Table 5.6, Rule 1} \\
 &= - \frac{d}{dx} \int_4^{1+3x^2} \frac{1}{2+e^t} dt \\
 &= - \frac{1}{2+e^{(1+3x^2)}} \frac{d}{dx} (1+3x^2) && \text{Eq. (2) and the Chain Rule} \\
 &= - \frac{6x}{2+e^{(1+3x^2)}}
 \end{aligned}$$

Proof of Theorem 4 We prove the Fundamental Theorem, Part 1, by applying the definition of the derivative directly to the function $F(x)$, when x and $x+h$ are in (a, b) . This means writing out the difference quotient

$$\frac{F(x+h) - F(x)}{h} \quad (3)$$

and showing that its limit as $h \rightarrow 0$ is the number $f(x)$. Doing so, we find that

$$\begin{aligned}
 F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt. && \text{Table 5.6, Rule 5}
 \end{aligned}$$

According to the Mean Value Theorem for Definite Integrals, there is some point c between x and $x+h$ where $f(c)$ equals the average value of f on the interval $[x, x+h]$. That is, there is some number c in $[x, x+h]$ such that

$$\frac{1}{h} \int_x^{x+h} f(t) dt = f(c). \quad (4)$$

As $h \rightarrow 0$, $x+h$ approaches x , which forces c to approach x also (because c is trapped between x and $x+h$). Since f is continuous at x , $f(c)$ therefore approaches $f(x)$:

$$\lim_{h \rightarrow 0} f(c) = f(x). \quad (5)$$

Hence we have shown that, for any x in (a, b) ,

$$\begin{aligned}
 F'(x) &= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt \\
 &= \lim_{h \rightarrow 0} f(c) && \text{Eq. (4)} \\
 &= f(x), && \text{Eq. (5)}
 \end{aligned}$$

and therefore F is differentiable at x . Since differentiability implies continuity, this also shows that F is continuous on the open interval (a, b) . To complete the proof, we just have to show that F is also continuous at $x = a$ and $x = b$. To do this, we make a very similar argument, except that at $x = a$ we need only consider the one-sided limit as $h \rightarrow 0^+$, and similarly at $x = b$ we need only consider $h \rightarrow 0^-$. This shows that F has a one-sided derivative at $x = a$ and at $x = b$, and therefore Theorem 1 in Section 3.2 implies that F is continuous at those two points. ■

Fundamental Theorem, Part 2 (The Evaluation Theorem)

We now come to the second part of the Fundamental Theorem of Calculus. This part describes how to evaluate definite integrals without having to calculate limits of Riemann sums. Instead we find and evaluate an antiderivative at the upper and lower limits of integration.

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2

If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Proof Part 1 of the Fundamental Theorem tells us that an antiderivative of f exists, namely

$$G(x) = \int_a^x f(t) dt.$$

Thus, if F is *any* antiderivative of f , then $F(x) = G(x) + C$ for some constant C for $a < x < b$ (by Corollary 2 of the Mean Value Theorem for Derivatives, Section 4.2). Since both F and G are continuous on $[a, b]$, we see that the equality $F(x) = G(x) + C$ also holds when $x = a$ and $x = b$ by taking one-sided limits (as $x \rightarrow a^+$ and $x \rightarrow b^-$).

Evaluating $F(b) - F(a)$, we have

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t) dt - \int_a^a f(t) dt \\ &= \int_a^b f(t) dt - 0 \\ &= \int_a^b f(t) dt. \end{aligned}$$

The Evaluation Theorem is important because it says that to calculate the definite integral of f over an interval $[a, b]$ we need do only two things:

1. Find an antiderivative F of f , and
2. Calculate the number $F(b) - F(a)$, which is equal to $\int_a^b f(x) dx$.

This process is much easier than computing Riemann sums and finding their limit. The power of the theorem follows from the realization that the definite integral, which is defined by a complicated process involving all of the values of the function f over $[a, b]$, can be found by knowing the values of *any* antiderivative F at only the two endpoints a and b . The usual notation for the difference $F(b) - F(a)$ is

$$F(x) \Big|_a^b \quad \text{or} \quad \left[F(x) \right]_a^b,$$

depending on whether F has one or more terms.

EXAMPLE 3 We calculate several definite integrals using the Evaluation Theorem, rather than by taking limits of Riemann sums.

$$\begin{aligned} \text{(a)} \quad \int_0^\pi \cos x dx &= \sin x \Big|_0^\pi & \frac{d}{dx} \sin x &= \cos x \\ &= \sin \pi - \sin 0 = 0 - 0 = 0 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_{-\pi/4}^0 \sec x \tan x \, dx &= \sec x \Big|_{-\pi/4}^0 & \frac{d}{dx} \sec x &= \sec x \tan x \\
 &= \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int_1^4 \left(\frac{3}{2} \sqrt{x} - \frac{4}{x^2} \right) dx &= \left[x^{3/2} + \frac{4}{x} \right]_1^4 & \frac{d}{dx} \left(x^{3/2} + \frac{4}{x} \right) &= \frac{3}{2} x^{1/2} - \frac{4}{x^2} \\
 &= \left[(4)^{3/2} + \frac{4}{4} \right] - \left[(1)^{3/2} + \frac{4}{1} \right] \\
 &= [8 + 1] - [5] = 4
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \int_1^2 \frac{1}{x} \, dx &= \ln|x| \Big|_1^2 & \frac{d}{dx} \ln|x| &= \frac{1}{x} \\
 &= \ln 2 - \ln 1 = \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad \int_0^1 \frac{dx}{x^2 + 1} &= \arctan x \Big|_0^1 & \frac{d}{dx} \arctan x &= \frac{1}{x^2 + 1} \\
 &= \arctan 1 - \arctan 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.
 \end{aligned}$$

Exercise 82 offers another proof of the Evaluation Theorem, bringing together the ideas of Riemann sums, the Mean Value Theorem, and the definition of the definite integral.

The Integral of a Rate

We can interpret Part 2 of the Fundamental Theorem in another way. If F is any antiderivative of f , then $F' = f$. The equation in the theorem can then be rewritten as

$$\int_a^b F'(x) \, dx = F(b) - F(a).$$

Now $F'(x)$ represents the rate of change of the function $F(x)$ with respect to x , so the last equation asserts that the integral of F' is just the *net change* in F as x changes from a to b . Formally, we have the following result.

THEOREM 5—The Net Change Theorem

The net change in a differentiable function $F(x)$ over an interval $a \leq x \leq b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x) \, dx. \quad (6)$$

EXAMPLE 4 Here are several interpretations of the Net Change Theorem.

(a) If $c(x)$ is the cost of producing x units of a certain commodity, then $c'(x)$ is the marginal cost (Section 3.4). From Theorem 5,

$$\int_{x_1}^{x_2} c'(x) \, dx = c(x_2) - c(x_1),$$

which is the cost of increasing production from x_1 units to x_2 units.

- (b) If an object with position function $s(t)$ moves along a coordinate line, its velocity is $v(t) = s'(t)$. Theorem 5 says that

$$\int_{t_1}^{t_2} v(t) dt = s(t_2) - s(t_1),$$

so the integral of velocity is the **displacement** over the time interval $t_1 \leq t \leq t_2$. On the other hand, the integral of the speed $|v(t)|$ is the **total distance traveled** over the time interval. This is consistent with our discussion in Section 5.1. ■

If we rearrange Equation (6) as

$$F(b) = F(a) + \int_a^b F'(x) dx,$$

we see that the Net Change Theorem also says that the final value of a function $F(x)$ over an interval $[a, b]$ equals its initial value $F(a)$ plus its net change over the interval. So if $v(t)$ represents the velocity function of an object moving along a coordinate line, this means that the object's final position $s(t_2)$ over a time interval $t_1 \leq t \leq t_2$ is its initial position $s(t_1)$ plus its net change in position along the line (see Example 4b).

EXAMPLE 5 Consider again our analysis of a heavy rock blown straight up from the ground by a dynamite blast (Example 2, Section 5.1). The velocity of the rock at any time t during its motion was given as $v(t) = 49 - 9.8t$ m/s.

- (a) Find the displacement of the rock during the time period $0 \leq t \leq 8$.
 (b) Find the total distance traveled during this time period.

Solution

- (a) From Example 4b, the displacement is the integral

$$\begin{aligned} \int_0^8 v(t) dt &= \int_0^8 (49 - 9.8t) dt = \left[49t - 4.9t^2 \right]_0^8 \\ &= (49)(8) - (4.9)(64) = 78.4. \end{aligned}$$

This means that the height of the rock is 78.4 m above the ground 8 s after the explosion, which agrees with our conclusion in Example 2, Section 5.1.

- (b) As we noted in Table 5.3, the velocity function $v(t)$ is positive over the time interval $[0, 5]$ and negative over the interval $[5, 8]$. Therefore, from Example 4b, the total distance traveled is the integral

$$\begin{aligned} \int_0^8 |v(t)| dt &= \int_0^5 |v(t)| dt + \int_5^8 |v(t)| dt \\ &= \int_0^5 (49 - 9.8t) dt - \int_5^8 (49 - 9.8t) dt \\ &= \left[49t - 4.9t^2 \right]_0^5 - \left[49t - 4.9t^2 \right]_5^8 \\ &= [(49)(5) - (4.9)(25)] - [(49)(8) - (4.9)(64) - ((49)(5) - (4.9)(25))] \\ &= 122.5 - (-44.1) = 166.6 \end{aligned}$$

Again, this calculation agrees with our conclusion in Example 2, Section 5.1. That is, the total distance of 166.6 m traveled by the rock during the time period $0 \leq t \leq 8$ is (i) the maximum height of 122.5 m it reached over the time interval $[0, 5]$ plus (ii) the additional distance of 44.1 m the rock fell over the time interval $[5, 8]$. ■

The Relationship Between Integration and Differentiation

The conclusions of the Fundamental Theorem tell us several things. Equation (2) can be rewritten as

$$\frac{d}{dx} \int_a^x f(t) dt = f(x),$$

which says that if you first integrate the function f and then differentiate the result, you get the function f back again. Likewise, replacing b by x and x by t in Equation (6) gives

$$\int_a^x F'(t) dt = F(x) - F(a),$$

so that if you first differentiate the function F and then integrate the result, you get the function F back (adjusted by an integration constant). In a sense, the processes of integration and differentiation are “inverses” of each other. The Fundamental Theorem also says that every continuous function f has an antiderivative F . It shows the importance of finding antiderivatives in order to evaluate definite integrals easily. Furthermore, it says that the differential equation $dy/dx = f(x)$ has a solution (namely, any of the functions $y = F(x) + C$) when f is a continuous function.

Total Area

Area is always a nonnegative quantity. The Riemann sum approximations contain terms such as $f(c_k) \Delta x_k$ that give the area of a rectangle when $f(c_k)$ is positive. When $f(c_k)$ is negative, then the product $f(c_k) \Delta x_k$ is the negative of the rectangle's area. When we add up such terms for a negative function, we get the negative of the area between the curve and the x -axis. If we then take the absolute value, we obtain the correct positive area.

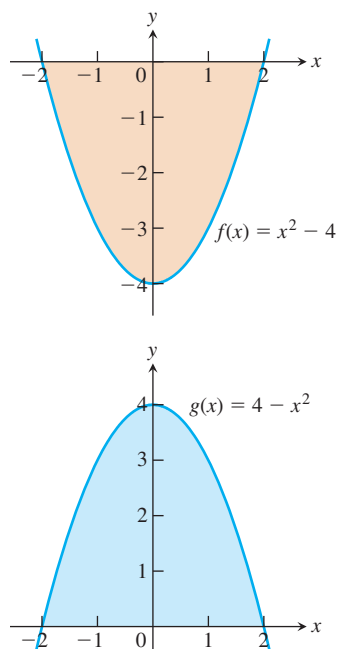


FIGURE 5.21 These graphs enclose the same amount of area with the x -axis, but the definite integrals of the two functions over $[-2, 2]$ differ in sign (Example 6).

EXAMPLE 6 Figure 5.21 shows the graph of $f(x) = x^2 - 4$ and its mirror image $g(x) = 4 - x^2$ reflected across the x -axis. For each function, compute

- (a) the definite integral over the interval $[-2, 2]$, and
- (b) the area between the graph and the x -axis over $[-2, 2]$.

Solution

$$(a) \int_{-2}^2 f(x) dx = \left[\frac{x^3}{3} - 4x \right]_{-2}^2 = \left(\frac{8}{3} - 8 \right) - \left(-\frac{8}{3} + 8 \right) = -\frac{32}{3},$$

and

$$\int_{-2}^2 g(x) dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3}.$$

- (b) In both cases, the area between the curve and the x -axis over $[-2, 2]$ is $32/3$ square units. Although the definite integral of $f(x)$ is negative, the area is still positive. ■

To compute the area of the region bounded by the graph of a function $y = f(x)$ and the x -axis when the function takes on both positive and negative values, we must be careful to break up the interval $[a, b]$ into subintervals on which the function doesn't change sign. Otherwise, we might get cancellation between positive and negative signed areas, leading to an incorrect total. The correct total area is obtained by adding the absolute value of the definite integral over each subinterval where $f(x)$ does not change sign. The term “area” will be taken to mean this *total area*.

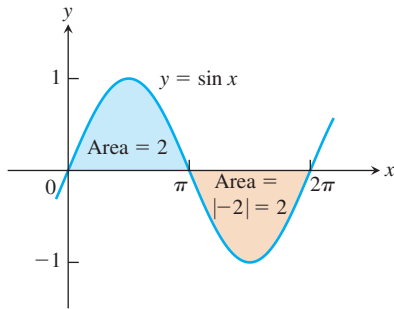


FIGURE 5.22 The total area between $y = \sin x$ and the x -axis for $0 \leq x \leq 2\pi$ is the sum of the absolute values of two integrals (Example 7).

EXAMPLE 7 Figure 5.22 shows the graph of the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

- (a) the definite integral of $f(x)$ over $[0, 2\pi]$,
- (b) the area between the graph of $f(x)$ and the x -axis over $[0, 2\pi]$.

Solution

- (a) The definite integral for $f(x) = \sin x$ is given by

$$\int_0^{2\pi} \sin x \, dx = -\cos x \Big|_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

The definite integral is zero because the portions of the graph above and below the x -axis make canceling contributions.

- (b) The area between the graph of $f(x)$ and the x -axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: the interval $[0, \pi]$ over which it is non-negative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2$$

$$\int_{\pi}^{2\pi} \sin x \, dx = -\cos x \Big|_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2$$

The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values,

$$\text{Area} = |2| + |-2| = 4.$$

Summary:

To find the area between the graph of $y = f(x)$ and the x -axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f .
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

EXAMPLE 8 Find the area of the region between the x -axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$.

Solution First find the zeros of f . Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2),$$

the zeros are $x = 0$, -1 , and 2 (Figure 5.23). The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \geq 0$, and $[0, 2]$, on which $f \leq 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^0 (x^3 - x^2 - 2x) \, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0 = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1 \right] = \frac{5}{12}$$

$$\int_0^2 (x^3 - x^2 - 2x) \, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_0^2 = \left[4 - \frac{8}{3} - 4 \right] - 0 = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals.

$$\text{Total enclosed area} = \frac{5}{12} + \left| -\frac{8}{3} \right| = \frac{37}{12}$$

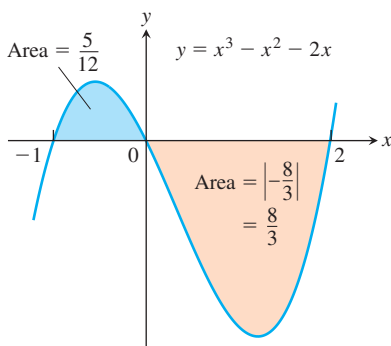


FIGURE 5.23 The region between the curve $y = x^3 - x^2 - 2x$ and the x -axis (Example 8).

EXERCISES 5.4

Evaluating Integrals

Evaluate the integrals in Exercises 1–34.

1. $\int_0^2 x(x-3) dx$
2. $\int_{-1}^1 (x^2 - 2x + 3) dx$
3. $\int_{-2}^2 (x+3)^2 dx$
4. $\int_{-1}^1 x^{299} dx$
5. $\int_1^4 \left(3x^2 - \frac{x^3}{4}\right) dx$
6. $\int_{-2}^3 (x^3 - 2x + 3) dx$
7. $\int_0^1 (x^2 + \sqrt{x}) dx$
8. $\int_1^{32} x^{-6/5} dx$
9. $\int_0^{\pi/3} 2 \sec^2 x dx$
10. $\int_0^{\pi} (1 + \cos x) dx$
11. $\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta d\theta$
12. $\int_0^{\pi/3} 4 \frac{\sin u}{\cos^2 u} du$
13. $\int_{\pi/2}^0 \frac{1 + \cos 2t}{2} dt$
14. $\int_{-\pi/3}^{\pi/3} \sin^2 t dt$
15. $\int_0^{\pi/4} \tan^2 x dx$
16. $\int_0^{\pi/6} (\sec x + \tan x)^2 dx$
17. $\int_0^{\pi/8} \sin 2x dx$
18. $\int_{-\pi/3}^{-\pi/4} \left(4 \sec^2 t + \frac{\pi}{t^2}\right) dt$
19. $\int_1^{-1} (r+1)^2 dr$
20. $\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4) dt$
21. $\int_{\sqrt{2}}^1 \left(\frac{u^7}{2} - \frac{1}{u^5}\right) du$
22. $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$
23. $\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2} ds$
24. $\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx$
25. $\int_{\pi/6}^{\pi/2} \frac{\sin 2x}{2 \sin x} dx$
26. $\int_0^{\pi/3} (\cos x + \sec x)^2 dx$
27. $\int_{-4}^4 |x| dx$
28. $\int_0^{\pi} \frac{1}{2} (\cos x + |\cos x|) dx$
29. $\int_0^{\ln 2} e^{3x} dx$
30. $\int_1^2 \left(\frac{1}{x} - e^{-x}\right) dx$
31. $\int_0^{1/2} \frac{4}{\sqrt{1-x^2}} dx$
32. $\int_0^{1/\sqrt{3}} \frac{dx}{1+4x^2}$
33. $\int_2^4 x^{\pi-1} dx$
34. $\int_{-1}^0 \pi^{x-1} dx$

In Exercises 35–38, guess an antiderivative for the integrand function. Validate your guess by differentiation, and then evaluate the given definite integral. (*Hint:* Keep the Chain Rule in mind when trying to guess an antiderivative. You will learn how to find such antiderivatives in the next section.)

35. $\int_0^1 x e^{x^2} dx$
36. $\int_1^2 \frac{\ln x}{x} dx$
37. $\int_2^5 \frac{x dx}{\sqrt{1+x^2}}$
38. $\int_0^{\pi/3} \sin^2 x \cos x dx$

Derivatives of Integrals

Find the derivatives in Exercises 39–44.

- a. by evaluating the integral and differentiating the result.
- b. by differentiating the integral directly.
39. $\frac{d}{dx} \int_0^{\sqrt{x}} \cos t dt$
40. $\frac{d}{dx} \int_1^{\sin x} 3t^2 dt$
41. $\frac{d}{dt} \int_0^{t^4} \sqrt{u} du$
42. $\frac{d}{d\theta} \int_0^{\tan \theta} \sec^2 y dy$
43. $\frac{d}{dx} \int_0^{x^3} e^{-t} dt$
44. $\frac{d}{dt} \int_0^{\sqrt{t}} \left(x^4 + \frac{3}{\sqrt{1-x^2}}\right) dx$

Find dy/dx in Exercises 45–56.

45. $y = \int_0^x \sqrt{1+t^2} dt$
46. $y = \int_1^x \frac{1}{t} dt, \quad x > 0$
47. $y = \int_{\sqrt{x}}^0 \sin t^2 dt$
48. $y = x \int_2^{x^2} \sin t^3 dt$
49. $y = \int_{-1}^x \frac{t^2}{t^2+4} dt - \int_3^x \frac{t^2}{t^2+4} dt$
50. $y = \left(\int_0^x (t^3+1)^{10} dt\right)^3$
51. $y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, \quad |x| < \frac{\pi}{2}$
52. $y = \int_{\tan x}^0 \frac{dt}{1+t^2}$
53. $y = \int_0^{e^{x^2}} \frac{1}{\sqrt{t}} dt$
54. $y = \int_{2^x}^1 \sqrt[3]{t} dt$
55. $y = \int_0^{\arcsin x} \cos t dt$
56. $y = \int_{-1}^{x^{1/\pi}} \arcsin t dt$

Area

In Exercises 57–60, find the total area between the region and the x -axis.

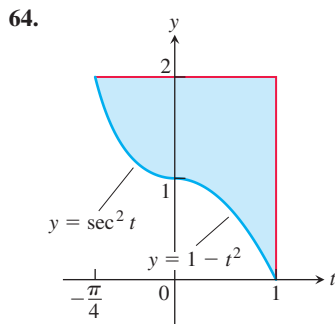
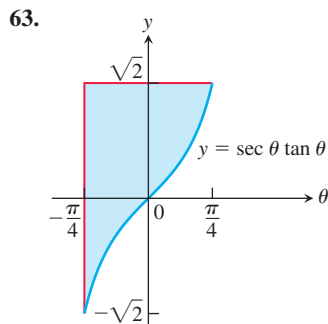
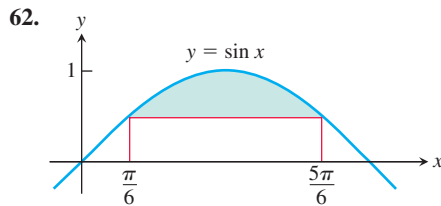
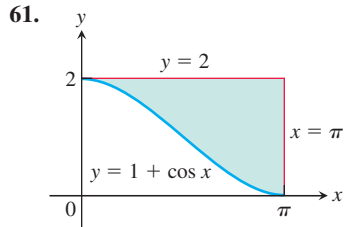
57. $y = -x^2 - 2x, \quad -3 \leq x \leq 2$

58. $y = 3x^2 - 3, \quad -2 \leq x \leq 2$

59. $y = x^3 - 3x^2 + 2x, \quad 0 \leq x \leq 2$

60. $y = x^{1/3} - x, \quad -1 \leq x \leq 8$

Find the areas of the shaded regions in Exercises 61–64.



Initial Value Problems

Each of the following functions solves one of the initial value problems in Exercises 65–68. Which function solves which problem? Give brief reasons for your answers.

a. $y = \int_1^x \frac{1}{t} dt - 3$ b. $y = \int_0^x \sec t dt + 4$

c. $y = \int_{-1}^x \sec t dt + 4$ d. $y = \int_{\pi}^x \frac{1}{t} dt - 3$

65. $\frac{dy}{dx} = \frac{1}{x}, \quad y(\pi) = -3$ 66. $y' = \sec x, \quad y(-1) = 4$

67. $y' = \sec x, \quad y(0) = 4$ 68. $y' = \frac{1}{x}, \quad y(1) = -3$

Express the solutions of the initial value problems in Exercises 69 and 70 in terms of integrals.

69. $\frac{dy}{dx} = \sec x, \quad y(2) = 3$ 70. $\frac{dy}{dx} = \sqrt{1+x^2}, \quad y(1) = -2$

For Exercises 71 and 72, find a function f satisfying each equation.

71. $\int_2^x \sqrt{f(t)} dt = x \ln x$ 72. $f(x) = e^2 + \int_1^x f(t) dt$

Theory and Examples

73. Archimedes' area formula for parabolic arches Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - (4h/b^2)x^2$, $-b/2 \leq x \leq b/2$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x -axis.

74. Show that if k is a positive constant, then the area between the x -axis and one arch of the curve $y = \sin kx$ is $2/k$.

75. Cost from marginal cost The marginal cost of printing a poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find $c(100) - c(1)$, the cost of printing posters 2–100.

In Exercises 76–78, guess an antiderivative and validate your guess by differentiation. (*Hint:* Keep the Chain Rule in mind when trying to guess an antiderivative. You will learn how to find such antiderivatives in the next section.)

76. Revenue from marginal revenue Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - 2/(x+1)^2,$$

where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

77. The temperature $T(^{\circ}\text{C})$ of a room at time t minutes is given by

$$T = 30 - 2\sqrt{25-t} \quad \text{for } 0 \leq t \leq 25.$$

a. Find the room's temperature when $t = 0$, $t = 16$, and $t = 25$.

b. Find the room's average temperature for $0 \leq t \leq 25$.

78. The height $H(\text{m})$ of a palm tree after growing for t years is given by

$$H = 0.3 - \sqrt{t+1} + 1.5t^{1/3} \quad \text{for } 0 \leq t \leq 8.$$

a. Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

b. Find the tree's average height for $0 \leq t \leq 8$.

79. Suppose that $\int_1^x f(t) dt = x^2 - 2x + 1$. Find $f(x)$.

80. Find $f(4)$ if $\int_0^x f(t) dt = x \cos \pi x$.

81. Find the linearization of

$$f(x) = 2 - \int_2^{x+1} \frac{9}{1+t} dt$$

at $x = 1$.

82. Find the linearization of

$$g(x) = 3 + \int_1^{x^2} \sec(t-1) dt$$

at $x = -1$.

83. Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t) dt?$$

Give reasons for your answers.

- g is a differentiable function of x .
- g is a continuous function of x .
- The graph of g has a horizontal tangent line at $x = 1$.
- g has a local maximum at $x = 1$.
- g has a local minimum at $x = 1$.
- The graph of g has an inflection point at $x = 1$.
- The graph of dg/dx crosses the x -axis at $x = 1$.

84. Another proof of the Evaluation Theorem

- a. Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$ be any partition of $[a, b]$, and let F be any antiderivative of f . Show that

$$F(b) - F(a) = \sum_{i=1}^n [F(x_i) - F(x_{i-1})].$$

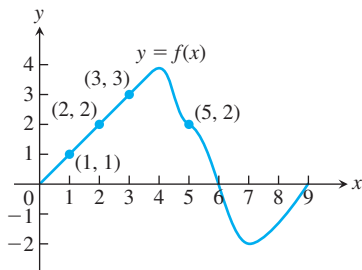
- b. Apply the Mean Value Theorem to each term to show that $F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$ for some c_i in the interval (x_{i-1}, x_i) . Then show that $F(b) - F(a)$ is a Riemann sum for f on $[a, b]$.
- c. From part (b) and the definition of the definite integral, show that

$$F(b) - F(a) = \int_a^b f(x) dx.$$

85. Suppose that f is the differentiable function shown in the accompanying graph and that the position at time t (in s) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x) dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



- What is the particle's velocity at time $t = 5$?
- Is the acceleration of the particle at time $t = 5$ positive or negative?
- What is the particle's position at time $t = 3$?
- At what time during the first 9 seconds does s have its largest value?
- Approximately when is the acceleration zero?
- When is the particle moving toward the origin? Away from the origin?

- g. On which side of the origin does the particle lie at time $t = 9$?

86. Find $\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} \int_1^x \frac{dt}{\sqrt{t}}$.

COMPUTER EXPLORATIONS

In Exercises 87–90, let $F(x) = \int_a^x f(t) dt$ for the specified function f and interval $[a, b]$. Use a CAS to perform the following steps and answer the questions posed.

- Plot the functions f and F together over $[a, b]$.
- Solve the equation $F'(x) = 0$. What can you see to be true about the graphs of f and F at points where $F'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem coupled with information provided by the first derivative? Explain your answer.
- Over what intervals (approximately) is the function F increasing? Decreasing? What is true about f over those intervals?
- Calculate the derivative f' and plot it together with F . What can you see to be true about the graph of F at points where $f'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem? Explain your answer.

87. $f(x) = x^3 - 4x^2 + 3x$, $[0, 4]$

88. $f(x) = 2x^4 - 17x^3 + 46x^2 - 43x + 12$, $\left[0, \frac{9}{2}\right]$

89. $f(x) = \sin 2x \cos \frac{x}{3}$, $[0, 2\pi]$

90. $f(x) = x \cos \pi x$, $[0, 2\pi]$

In Exercises 91–94, let $F(x) = \int_a^{u(x)} f(t) dt$ for the specified a , u , and f . Use a CAS to perform the following steps and answer the questions posed.

- Find the domain of F .
- Calculate $F'(x)$ and determine its zeros. For what points in its domain is F increasing? Decreasing?
- Calculate $F''(x)$ and determine its zero. Identify the local extrema and the points of inflection of F .
- Using the information from parts (a)–(c), draw a rough hand-sketched $y = F(x)$ over its domain. Then graph $F(x)$ on your CAS to support your sketch.

91. $a = 1$, $u(x) = x^2$, $f(x) = \sqrt{1 - x^2}$

92. $a = 0$, $u(x) = x^2$, $f(x) = \sqrt{1 - x^2}$

93. $a = 0$, $u(x) = 1 - x$, $f(x) = x^2 - 2x - 3$

94. $a = 0$, $u(x) = 1 - x^2$, $f(x) = x^2 - 2x - 3$

In Exercises 95 and 96, assume that f is continuous and $u(x)$ is twice-differentiable.

95. Calculate $\frac{d}{dx} \int_a^{u(x)} f(t) dt$ and check your answer using a CAS.

96. Calculate $\frac{d^2}{dx^2} \int_a^{u(x)} f(t) dt$ and check your answer using a CAS.

5.5 Indefinite Integrals and the Substitution Method

The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed directly if we can find an antiderivative of the function. In Section 4.8 we defined the **indefinite integral** of the function f with respect to x as the set of *all* antiderivatives of f , symbolized by $\int f(x) dx$. Since any two antiderivatives of f differ by a constant, the indefinite integral $\int$ notation means that for any antiderivative F of f ,

$$\int f(x) dx = F(x) + C,$$

where C is any arbitrary constant. The connection between antiderivatives and the definite integral stated in the Fundamental Theorem now explains this notation:

$$\begin{aligned}\int_a^b f(x) dx &= F(b) - F(a) = [F(b) + C] - [F(a) + C] \\ &= \left[F(x) + C \right]_a^b = \left[\int f(x) dx \right]_a^b.\end{aligned}$$

When finding the indefinite integral of a function f , remember that it always includes an arbitrary constant C .

We must keep in mind the difference between definite and indefinite integrals. A definite integral $\int_a^b f(x) dx$ is a *number*. An indefinite integral $\int f(x) dx$ is a *function* plus an arbitrary constant C .

So far, we have only been able to find antiderivatives of functions that are clearly recognizable as derivatives. In this section we begin to develop more general techniques for finding antiderivatives of functions.

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x , and n is any number different from -1 , the Chain Rule tells us that

$$\frac{d}{dx} \left(\frac{u^{n+1}}{n+1} \right) = u^n \frac{du}{dx}.$$

From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n (du/dx)$. Therefore,

$$\int u^n \frac{du}{dx} dx = \frac{u^{n+1}}{n+1} + C. \quad (1)$$

The integral in Equation (1) is equal to the simpler integral

$$\int u^n du = \frac{u^{n+1}}{n+1} + C,$$

which suggests that we can substitute the simpler expression du for $(du/dx) dx$ when computing an integral. Leibniz, one of the founders of calculus, had the insight that indeed this substitution could be done, leading to the *substitution method* for computing integrals. As with differentials, when computing integrals we have

$$du = \frac{du}{dx} dx.$$

EXAMPLE 1 Find the integral $\int (x^3 + x)^5 (3x^2 + 1) dx$.

Solution We set $u = x^3 + x$. Then

$$du = \frac{du}{dx} dx = (3x^2 + 1) dx,$$

so that by substitution we have

$$\begin{aligned}
 \int (x^3 + x)^5 (3x^2 + 1) dx &= \int u^5 du && \text{Let } u = x^3 + x, du = (3x^2 + 1) dx. \\
 &= \frac{u^6}{6} + C && \text{Integrate with respect to } u. \\
 &= \frac{(x^3 + x)^6}{6} + C && \text{Substitute } x^3 + x \text{ for } u.
 \end{aligned}$$

EXAMPLE 2 Find $\int \sqrt{2x + 1} dx$.

Solution The integral does not fit the formula

$$\int u^n du,$$

with $u = 2x + 1$ and $n = 1/2$, because

$$du = \frac{du}{dx} dx = 2 dx,$$

which is not precisely dx . The constant factor 2 is missing from the integral. However, we can introduce this factor after the integral sign if we compensate for it by introducing a factor of $1/2$ in front of the integral sign. So we write

$$\begin{aligned}
 \int \sqrt{2x + 1} dx &= \frac{1}{2} \int \sqrt{\underbrace{2x + 1}_u} \cdot \underbrace{2 dx}_{du} \\
 &= \frac{1}{2} \int u^{1/2} du && \text{Let } u = 2x + 1, du = 2 dx. \\
 &= \frac{1}{2} \frac{u^{3/2}}{3/2} + C && \text{Integrate with respect to } u. \\
 &= \frac{1}{3} (2x + 1)^{3/2} + C. && \text{Substitute } 2x + 1 \text{ for } u.
 \end{aligned}$$

The substitutions in Examples 1 and 2 are instances of the following general rule.

THEOREM 6—The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I , and f is continuous on I , then

$$\int f(g(x)) \cdot g'(x) dx = \int f(u) du.$$

Proof By the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f , because

$$\begin{aligned}
 \frac{d}{dx} F(g(x)) &= F'(g(x)) \cdot g'(x) && \text{Chain Rule} \\
 &= f(g(x)) \cdot g'(x). && F' = f
 \end{aligned}$$

If we make the substitution $u = g(x)$, then

$$\begin{aligned}
 \int f(g(x)) g'(x) dx &= \int \frac{d}{dx} F(g(x)) dx \\
 &= F(g(x)) + C && \text{Theorem 8 in Chapter 4} \\
 &= F(u) + C && u = g(x) \\
 &= \int F'(u) du && \text{Theorem 8 in Chapter 4} \\
 &= \int f(u) du. && F' = f
 \end{aligned}$$

The use of the variable u in the Substitution Rule is traditional (sometimes it is referred to as u -substitution), but any letter can be used, such as v, t, θ and so forth. The rule provides a method for evaluating an integral of the form $\int f(g(x)) g'(x) dx$, given that the conditions of Theorem 6 are satisfied. The primary challenge is deciding what expression involving x to substitute for in the integrand. The following examples give helpful ideas.

The Substitution Method to evaluate $\int f(g(x)) g'(x) dx$

1. Substitute $u = g(x)$ and $du = (du/dx) dx = g'(x) dx$ to obtain $\int f(u) du$.
2. Integrate with respect to u .
3. Replace u by $g(x)$.

EXAMPLE 3 Find $\int \sec^2(5x + 1) \cdot 5 dx$.

Solution We substitute $u = 5x + 1$ and $du = 5 dx$. Then

$$\begin{aligned}
 \int \sec^2(5x + 1) \cdot 5 dx &= \int \sec^2 u du && \text{Let } u = 5x + 1, du = 5 dx. \\
 &= \tan u + C && \frac{d}{du} \tan u = \sec^2 u \\
 &= \tan(5x + 1) + C. && \text{Substitute } 5x + 1 \text{ for } u.
 \end{aligned}$$

EXAMPLE 4 Find $\int \cos(7\theta + 3) d\theta$.

Solution We let $u = 7\theta + 3$ so that $du = 7 d\theta$. There is a factor of 7 in this formula for du , but there is no corresponding 7 preceding $d\theta$ in the integral. We can compensate for this by multiplying and dividing by 7, using the same procedure as in Example 2. Then

$$\begin{aligned}
 \int \cos(7\theta + 3) d\theta &= \frac{1}{7} \int \cos(7\theta + 3) \cdot 7 d\theta && \text{Place factor } 1/7 \text{ in front of integral.} \\
 &= \frac{1}{7} \int \cos u du && \text{Substitute } u = 7\theta + 3, du = 7 d\theta. \\
 &= \frac{1}{7} \sin u + C && \text{Integrate.} \\
 &= \frac{1}{7} \sin(7\theta + 3) + C. && \text{Replace } u \text{ by } 7\theta + 3.
 \end{aligned}$$

There is another approach to this problem. With $u = 7\theta + 3$ and $du = 7 d\theta$ as before, we solve for $d\theta$ to obtain $d\theta = (1/7) du$. Then the integral becomes

$$\begin{aligned}\int \cos(7\theta + 3) d\theta &= \int \cos u \cdot \frac{1}{7} du && \text{Substitute } u = 7\theta + 3, du = 7 d\theta, \\ &= \frac{1}{7} \sin u + C && \text{and } d\theta = (1/7) du. \\ &= \frac{1}{7} \sin(7\theta + 3) + C. && \text{Integrate.} \\ &&& \text{Replace } u \text{ by } 7\theta + 3.\end{aligned}$$

We can verify this solution by differentiating and checking that we obtain the original function $\cos(7\theta + 3)$. ■

EXAMPLE 5 Sometimes we observe that a power of x appears in the integrand that is one less than the power of x appearing in the argument of a function we want to integrate. This observation immediately suggests we try a substitution for the higher power of x . For example, in the integral below we see that x^3 appears as the exponent of one factor, and this factor is multiplied by x^2 . This suggests trying the substitution $u = x^3$.

$$\begin{aligned}\int x^2 e^{x^3} dx &= \int e^{x^3} \cdot x^2 dx \\ &= \int e^u \cdot \frac{1}{3} du && \text{Substitute } u = x^3, du = 3x^2 dx, \\ &= \frac{1}{3} \int e^u du && (1/3) du = x^2 dx. \\ &= \frac{1}{3} e^u + C && \text{Integrate with respect to } u. \\ &= \frac{1}{3} e^{x^3} + C && \text{Replace } u \text{ by } x^3.\end{aligned}$$
■

HISTORICAL BIOGRAPHY

George David Birkhoff
(1884–1944)

Birkhoff attended Harvard and the University of Chicago. He received his PhD from Chicago in 1907 for his dissertation on differential equations. He also worked on the four-color problem (colors required to produce a map) and applying mathematics to aesthetics in art, poetry, and music.

To know more, visit the companion Website.

It may happen that an extra factor of x appears in the integrand when we try a substitution $u = g(x)$. In that case, it may be possible to solve the equation $u = g(x)$ for x in terms of u . Replacing the extra factor of x with that expression may then result in an integral that we can evaluate. Here is an example of this situation.

EXAMPLE 6 Evaluate $\int x\sqrt{2x+1} dx$.

Solution Our previous experience with the integral in Example 2 suggests the substitution $u = 2x + 1$ with $du = 2 dx$. Then

$$\sqrt{2x+1} dx = \frac{1}{2} \sqrt{u} du.$$

However, in this example the integrand contains an extra factor of x that multiplies the factor $\sqrt{2x+1}$. To adjust for this, we solve the substitution equation $u = 2x + 1$ for x to obtain $x = (u - 1)/2$ and find that

$$x\sqrt{2x+1} dx = \frac{1}{2}(u - 1) \cdot \frac{1}{2} \sqrt{u} du.$$

The integration now becomes

$$\begin{aligned}
 \int x\sqrt{2x+1} \, dx &= \frac{1}{4} \int (u-1)\sqrt{u} \, du = \frac{1}{4} \int (u-1)u^{1/2} \, du && \text{Substitute.} \\
 &= \frac{1}{4} \int (u^{3/2} - u^{1/2}) \, du && \text{Multiply terms by } u^{1/2}. \\
 &= \frac{1}{4} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C && \text{Integrate.} \\
 &= \frac{1}{10} (2x+1)^{5/2} - \frac{1}{6} (2x+1)^{3/2} + C. && \text{Replace } u \text{ by } 2x+1. \quad \blacksquare
 \end{aligned}$$

EXAMPLE 7 Sometimes we can use trigonometric identities to transform an integral we do not know how to evaluate into one that we can evaluate using the Substitution Rule.

$$\begin{aligned}
 \text{(a)} \quad \int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx && \sin^2 x = \frac{1 - \cos 2x}{2} \\
 &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\
 &= \frac{1}{2} x - \frac{1}{2} \frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C \\
 \text{(b)} \quad \int \cos^2 x \, dx &= \int \frac{1 + \cos 2x}{2} \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C && \cos^2 x = \frac{1 + \cos 2x}{2} \\
 \text{(c)} \quad \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = \int \frac{-du}{u} && u = \cos x, \, du = -\sin x \, dx \\
 &= -\ln |u| + C = -\ln |\cos x| + C \\
 &= \ln \frac{1}{|\cos x|} + C = \ln |\sec x| + C && \text{Reciprocal Rule} \quad \blacksquare
 \end{aligned}$$

EXAMPLE 8 An integrand may require some algebraic manipulation before the substitution method can be applied. This example gives two integrals for which we simplify by multiplying the integrand by an algebraic form equal to 1 before attempting a substitution.

$$\begin{aligned}
 \text{(a)} \quad \int \frac{dx}{e^x + e^{-x}} &= \int \frac{e^x \, dx}{e^{2x} + 1} && \text{Multiply by } (e^x/e^x) = 1. \\
 &= \int \frac{du}{u^2 + 1} && \text{Substitute } u = e^x, \, u^2 = e^{2x}, \\
 &= \arctan u + C && du = e^x \, dx. \\
 &= \arctan(e^x) + C && \text{Integrate with respect to } u. \\
 &&& \text{Replace } u \text{ by } e^x. \\
 \text{(b)} \quad \int \sec x \, dx &= \int (\sec x)(1) \, dx \\
 &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx && \frac{\sec x + \tan x}{\sec x + \tan x} \text{ is equal to } 1. \\
 &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\
 &= \int \frac{du}{u} && u = \sec x + \tan x, \\
 &= \ln |u| + C = \ln |\sec x + \tan x| + C. && du = (\sec^2 x + \sec x \tan x) \, dx \quad \blacksquare
 \end{aligned}$$

The integrals of $\cot x$ and $\csc x$ are computed in a way similar to the integrals of $\tan x$ and $\sec x$ in Examples 7c and 8b (see Exercises 71 and 72). We summarize the results for these four basic trigonometric integrals here.

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\begin{aligned}\int \tan x \, dx &= \ln |\sec x| + C & \int \sec x \, dx &= \ln |\sec x + \tan x| + C \\ \int \cot x \, dx &= \ln |\sin x| + C & \int \csc x \, dx &= -\ln |\csc x + \cot x| + C\end{aligned}$$

Trying Different Substitutions

The success of the substitution method depends on finding a substitution that changes an integral we cannot directly evaluate into one that we can. Finding the right substitution gets easier with practice and experience. If your first substitution fails, try another substitution, possibly coupled with other algebraic or trigonometric simplifications to the integrand. Several more complicated types of substitutions will be studied in Chapter 8.

EXAMPLE 9 Evaluate $\int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}}$.

Solution We use the substitution method of integration as an exploratory tool. First we substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$, or we might even press our luck and take u to be the entire cube root. In this example both substitutions turn out to be successful, but that is not always the case. If one substitution does not help, a different substitution may work instead.

Method 1: Substitute $u = z^2 + 1$.

$$\begin{aligned}\int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{du}{u^{1/3}} && \text{Let } u = z^2 + 1, \, du = 2z \, dz. \\ &= \int u^{-1/3} \, du && \text{In the form } \int u^n \, du \\ &= \frac{u^{2/3}}{2/3} + C && \text{Integrate.} \\ &= \frac{3}{2} u^{2/3} + C \\ &= \frac{3}{2} (z^2 + 1)^{2/3} + C && \text{Replace } u \text{ by } z^2 + 1.\end{aligned}$$

Method 2: Substitute $u = \sqrt[3]{z^2 + 1}$ instead.

$$\begin{aligned}\int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{3u^2 \, du}{u} && \text{Let } u = \sqrt[3]{z^2 + 1}, \\ &= 3 \int u \, du && u^2 = z^2 + 1, \, 3u^2 \, du = 2z \, dz. \\ &= 3 \cdot \frac{u^2}{2} + C && \text{Integrate.} \\ &= \frac{3}{2} (z^2 + 1)^{2/3} + C && \text{Replace } u \text{ by } (z^2 + 1)^{1/3}.\end{aligned}$$

EXERCISES 5.5

Evaluating Indefinite Integrals

In Exercises 1–16, make the given substitutions to evaluate the indefinite integrals.

1. $\int 2(2x + 4)^5 dx$, $u = 2x + 4$
2. $\int 7\sqrt{7x - 1} dx$, $u = 7x - 1$
3. $\int 2x(x^2 + 5)^{-4} dx$, $u = x^2 + 5$
4. $\int \frac{4x^3}{(x^4 + 1)^2} dx$, $u = x^4 + 1$
5. $\int (3x + 2)(3x^2 + 4x)^4 dx$, $u = 3x^2 + 4x$
6. $\int \frac{(1 + \sqrt{x})^{1/3}}{\sqrt{x}} dx$, $u = 1 + \sqrt{x}$
7. $\int \sin 3x dx$, $u = 3x$
8. $\int x \sin(2x^2) dx$, $u = 2x^2$
9. $\int \sec 2t \tan 2t dt$, $u = 2t$
10. $\int \left(1 - \cos \frac{t}{2}\right)^2 \sin \frac{t}{2} dt$, $u = 1 - \cos \frac{t}{2}$
11. $\int \frac{9r^2 dr}{\sqrt{1 - r^3}}$, $u = 1 - r^3$
12. $\int 12(y^4 + 4y^2 + 1)^2 (y^3 + 2y) dy$, $u = y^4 + 4y^2 + 1$
13. $\int \sqrt{x} \sin^2(x^{3/2} - 1) dx$, $u = x^{3/2} - 1$
14. $\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) dx$, $u = \frac{1}{x}$
15. $\int \csc^2 2\theta \cot 2\theta d\theta$
 - a. Using $u = \cot 2\theta$
 - b. Using $u = \csc 2\theta$
16. $\int \frac{dx}{\sqrt{5x + 8}}$
 - a. Using $u = 5x + 8$
 - b. Using $u = \sqrt{5x + 8}$

Evaluate the integrals in Exercises 17–66.

17. $\int \sqrt{3 - 2s} ds$
18. $\int \frac{1}{\sqrt{5s + 4}} ds$
19. $\int \theta \sqrt{1 - \theta^2} d\theta$
20. $\int 3y \sqrt{7 - 3y^2} dy$
21. $\int \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} dx$
22. $\int \sqrt{\sin x} \cos^3 x dx$
23. $\int \sec^2(3x + 2) dx$
24. $\int \tan^2 x \sec^2 x dx$
25. $\int \sin^5 \frac{x}{3} \cos \frac{x}{3} dx$
26. $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} dx$
27. $\int r^2 \left(\frac{r^3}{18} - 1\right)^5 dr$
28. $\int r^4 \left(7 - \frac{r^5}{10}\right)^3 dr$
29. $\int x^{1/2} \sin(x^{3/2} + 1) dx$
30. $\int \csc\left(\frac{v - \pi}{2}\right) \cot\left(\frac{v - \pi}{2}\right) dv$
31. $\int \frac{\sin(2t + 1)}{\cos^2(2t + 1)} dt$
32. $\int \frac{\sec z \tan z}{\sqrt{\sec z}} dz$
33. $\int \frac{1}{t^2} \cos\left(\frac{1}{t} - 1\right) dt$
34. $\int \frac{1}{\sqrt{t}} \cos(\sqrt{t} + 3) dt$
35. $\int \frac{1}{\theta^2} \sin \frac{1}{\theta} \cos \frac{1}{\theta} d\theta$
36. $\int \frac{\cos \sqrt{\theta}}{\sqrt{\theta} \sin^2 \sqrt{\theta}} d\theta$
37. $\int \frac{x}{\sqrt{1 + x}} dx$
38. $\int \sqrt{\frac{x - 1}{x^5}} dx$
39. $\int \frac{1}{x^2} \sqrt{2 - \frac{1}{x}} dx$
40. $\int \frac{1}{x^3} \sqrt{\frac{x^2 - 1}{x^2}} dx$
41. $\int \sqrt{\frac{x^3 - 3}{x^{11}}} dx$
42. $\int \sqrt{\frac{x^4}{x^3 - 1}} dx$
43. $\int x(x - 1)^{10} dx$
44. $\int x \sqrt{4 - x} dx$
45. $\int (x + 1)^2 (1 - x)^5 dx$
46. $\int (x + 5)(x - 5)^{1/3} dx$
47. $\int x^3 \sqrt{x^2 + 1} dx$
48. $\int 3x^5 \sqrt{x^3 + 1} dx$
49. $\int \frac{x}{(x^2 - 4)^3} dx$
50. $\int \frac{x}{(2x - 1)^{2/3}} dx$
51. $\int (\cos x) e^{\sin x} dx$
52. $\int (\sin 2\theta) e^{\sin^2 \theta} d\theta$
53. $\int \frac{1}{\sqrt{x} e^{-\sqrt{x}}} \sec^2(e^{\sqrt{x}} + 1) dx$
54. $\int \frac{1}{x^2} e^{1/x} \sec(1 + e^{1/x}) \tan(1 + e^{1/x}) dx$
55. $\int \frac{dx}{x \ln x}$
56. $\int \frac{\ln \sqrt{t}}{t} dt$
57. $\int \frac{dz}{1 + e^z}$
58. $\int \frac{dx}{x \sqrt{x^4 - 1}}$
59. $\int \frac{5}{9 + 4r^2} dr$
60. $\int \frac{1}{\sqrt{e^{2\theta} - 1}} d\theta$

61. $\int \frac{e^{\arcsin x} dx}{\sqrt{1-x^2}}$

62. $\int \frac{e^{\arccos x} dx}{\sqrt{1-x^2}}$

63. $\int \frac{(\arcsin x)^2 dx}{\sqrt{1-x^2}}$

64. $\int \frac{\sqrt{\arctan x} dx}{1+x^2}$

65. $\int \frac{dy}{(\arctan y)(1+y^2)}$

66. $\int \frac{dy}{(\arcsin y)\sqrt{1-y^2}}$

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 67 and 68.

67. $\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx$

a. $u = \tan x$, followed by $v = u^3$, then by $w = 2 + v$

b. $u = \tan^3 x$, followed by $v = 2 + u$

c. $u = 2 + \tan^3 x$

68. $\int \sqrt{1 + \sin^2(x-1)} \sin(x-1) \cos(x-1) dx$

a. $u = x - 1$, followed by $v = \sin u$, then by $w = 1 + v^2$

b. $u = \sin(x - 1)$, followed by $v = 1 + u^2$

c. $u = 1 + \sin^2(x - 1)$

Evaluate the integrals in Exercises 69 and 70.

69. $\int \frac{(2r-1) \cos \sqrt{3(2r-1)^2 + 6}}{\sqrt{3(2r-1)^2 + 6}} dr$

70. $\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta} \cos^3 \sqrt{\theta}} d\theta$

71. Find the integral of $\cot x$ using a substitution like that in Example 7c.

72. Find the integral of $\csc x$ by multiplying by an appropriate form equal to 1, as in Example 8b.

Initial Value Problems

Solve the initial value problems in Exercises 73–78.

73. $\frac{ds}{dt} = 12t(3t^2 - 1)^3, \quad s(1) = 3$

74. $\frac{dy}{dx} = 4x(x^2 + 8)^{-1/3}, \quad y(0) = 0$

75. $\frac{ds}{dt} = 8 \sin^2\left(t + \frac{\pi}{12}\right), \quad s(0) = 8$

76. $\frac{dr}{d\theta} = 3 \cos^2\left(\frac{\pi}{4} - \theta\right), \quad r(0) = \frac{\pi}{8}$

77. $\frac{d^2s}{dt^2} = -4 \sin\left(2t - \frac{\pi}{2}\right), \quad s'(0) = 100, \quad s(0) = 0$

78. $\frac{d^2y}{dx^2} = 4 \sec^2 2x \tan 2x, \quad y'(0) = 4, \quad y(0) = -1$

79. The velocity of a particle moving back and forth on a line is $v = ds/dt = 6 \sin 2t$ m/s for all t . If $s = 0$ when $t = 0$, find the value of s when $t = \pi/2$ s.

80. The acceleration of a particle moving back and forth on a line is $a = d^2s/dt^2 = \pi^2 \cos \pi t$ m/s² for all t . If $s = 0$ and $v = 8$ m/s when $t = 0$, find s when $t = 1$ s.

5.6 Definite Integral Substitutions and the Area Between Curves

There are two methods for evaluating a definite integral by substitution. One method is to find an antiderivative using substitution and then to evaluate the definite integral by applying the Evaluation Theorem. The other method extends the process of substitution directly to *definite* integrals by changing the limits of integration. We will use these methods to compute the area between two curves.

The Substitution Formula

The following formula shows how the limits of integration change when we apply a substitution to an integral.

THEOREM 7—Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Proof Let F denote any antiderivative of f . Then

$$\begin{aligned}
 \int_a^b f(g(x)) \cdot g'(x) dx &= F(g(x)) \Big|_{x=a}^{x=b} & \frac{d}{dx} F(g(x)) &= F'(g(x)) g'(x) \\
 &= F(g(b)) - F(g(a)) & &= f(g(x)) g'(x) \\
 &= F(u) \Big|_{u=g(a)}^{u=g(b)} \\
 &= \int_{g(a)}^{g(b)} f(u) du. & \text{Fundamental Theorem, Part 2} & \blacksquare
 \end{aligned}$$

To use Theorem 7, we make the same u -substitution $u = g(x)$ and $du = g'(x) dx$ that we would use to evaluate the corresponding indefinite integral. We then integrate the transformed integral with respect to u from the value $g(a)$ (the value of u at $x = a$) to the value $g(b)$ (the value of u at $x = b$).

EXAMPLE 1 Evaluate $\int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx$.

Solution We will show how to evaluate the integral using Theorem 7, and how to evaluate it using the original limits of integration.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 7.

$$\begin{aligned}
 \int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx &= \int_0^2 \sqrt{u} du & \text{Let } u = x^3 + 1, du = 3x^2 dx. \\
 & & \text{When } x = -1, u = (-1)^3 + 1 = 0. \\
 & & \text{When } x = 1, u = (1)^3 + 1 = 2. \\
 &= \frac{2}{3} u^{3/2} \Big|_0^2 & \text{Evaluate the new definite integral.} \\
 &= \frac{2}{3} [2^{3/2} - 0^{3/2}] = \frac{2}{3} [2\sqrt{2}] = \frac{4\sqrt{2}}{3}
 \end{aligned}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x , and use the original x -limits.

$$\begin{aligned}
 \int 3x^2 \sqrt{x^3 + 1} dx &= \int \sqrt{u} du & \text{Let } u = x^3 + 1, du = 3x^2 dx. \\
 &= \frac{2}{3} u^{3/2} + C & \text{Integrate with respect to } u. \\
 &= \frac{2}{3} (x^3 + 1)^{3/2} + C & \text{Replace } u \text{ by } x^3 + 1. \\
 \int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx &= \frac{2}{3} (x^3 + 1)^{3/2} \Big|_{-1}^1 & \text{Use the integral just found, with} \\
 & & \text{limits of integration for } x. \\
 &= \frac{2}{3} [(1^3 + 1)^{3/2} - ((-1)^3 + 1)^{3/2}] \\
 &= \frac{2}{3} [2^{3/2} - 0^{3/2}] = \frac{2}{3} [2\sqrt{2}] = \frac{4\sqrt{2}}{3} \quad \blacksquare
 \end{aligned}$$

Which method is better—evaluating the transformed definite integral with transformed limits using Theorem 7, or transforming the integral, integrating, and transforming back to use the original limits of integration? In Example 1, the first method seems easier, but that is not always the case. Generally, it is best to know both methods and to use whichever one seems better at the time.

EXAMPLE 2 We use the method of transforming the limits of integration.

$$\begin{aligned}
 \text{(a)} \quad \int_{\pi/4}^{\pi/2} \cot \theta \csc^2 \theta \, d\theta &= \int_1^0 u \cdot (-du) \\
 &= -\int_1^0 u \, du \\
 &= -\left[\frac{u^2}{2}\right]_1^0 \\
 &= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}
 \end{aligned}$$

Let $u = \cot \theta$, $du = -\csc^2 \theta \, d\theta$,
 $-du = \csc^2 \theta \, d\theta$.
 When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$.
 When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$.

$$\begin{aligned}
 \text{(b)} \quad \int_{-\pi/4}^{\pi/4} \tan x \, dx &= \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x} \, dx \\
 &= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u} \\
 &= 0
 \end{aligned}$$

Let $u = \cos x$, $du = -\sin x \, dx$.
 When $x = -\pi/4$, $u = \sqrt{2}/2$.
 When $x = \pi/4$, $u = \sqrt{2}/2$.
 Zero width interval

Definite Integrals of Symmetric Functions

The Substitution Formula in Theorem 7 simplifies the calculation of definite integrals of even and odd functions (Section 1.1) over a symmetric interval $[-a, a]$ (Figure 5.24).

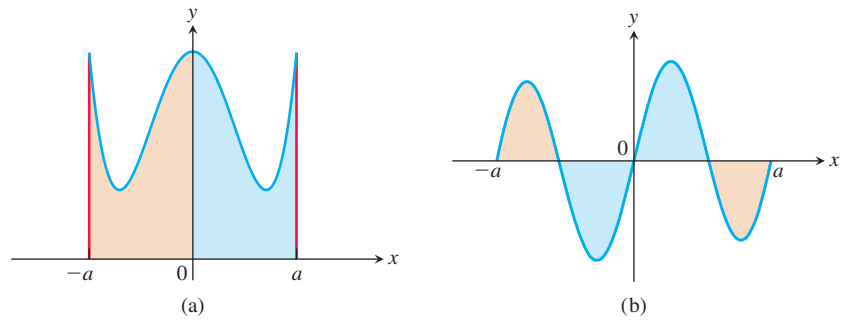


FIGURE 5.24 (a) For f an even function, the integral from $-a$ to a is twice the integral from 0 to a . (b) For f an odd function, the integral from $-a$ to a equals 0.

THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx$.

(b) If f is odd, then $\int_{-a}^a f(x) \, dx = 0$.

Proof of Part (a)

$$\begin{aligned}
 \int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx && \text{Additivity Rule for Definite Integrals} \\
 &= -\int_0^{-a} f(x) dx + \int_0^a f(x) dx && \text{Order of Integration Rule} \\
 &= -\int_0^a f(-u)(-du) + \int_0^a f(x) dx && \begin{array}{l} \text{Let } u = -x, du = -dx. \\ \text{When } x = 0, u = 0. \\ \text{When } x = -a, u = a. \end{array} \\
 &= \int_0^a f(-u) du + \int_0^a f(x) dx \\
 &= \int_0^a f(u) du + \int_0^a f(x) dx && \begin{array}{l} f \text{ is even, so} \\ f(-u) = f(u). \end{array} \\
 &= 2 \int_0^a f(x) dx
 \end{aligned}$$

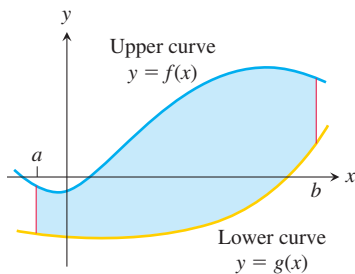


FIGURE 5.25 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

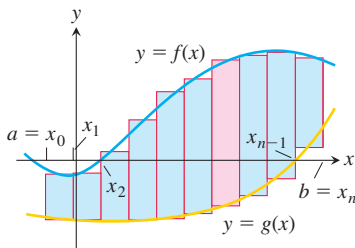


FIGURE 5.26 We approximate the region with rectangles perpendicular to the x -axis.

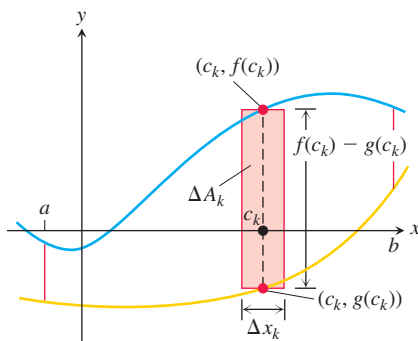


FIGURE 5.27 The area ΔA_k of the k th rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k .

The proof of part (b) is similar, and you are asked to give it in Exercise 120. ■

EXAMPLE 3 Evaluate $\int_{-2}^2 x^4 - 4x^2 + 6 dx$.

Solution Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

$$\begin{aligned}
 \int_{-2}^2 x^4 - 4x^2 + 6 dx &= 2 \int_0^2 x^4 - 4x^2 + 6 dx \\
 &= 2 \left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x \right]_0^2 \\
 &= 2 \left(\frac{32}{5} - \frac{32}{3} + 12 \right) = \frac{232}{15}.
 \end{aligned}$$

Areas Between Curves

Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$ (Figure 5.25). The region might accidentally have a shape whose area we could find with geometry, but if f and g are arbitrary continuous functions, we usually have to find the area by computing an integral.

To see what the integral should be, we first approximate the region with n vertical rectangles based on a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ (Figure 5.26). The area of the k th rectangle (Figure 5.27) is

$$\Delta A_k = \text{height} \times \text{width} = [f(c_k) - g(c_k)] \Delta x_k.$$

We then approximate the area of the region by adding the areas of the n rectangles:

$$A \approx \sum_{k=1}^n \Delta A_k = \sum_{k=1}^n [f(c_k) - g(c_k)] \Delta x_k. \quad \text{Riemann sum}$$

As $\|P\| \rightarrow 0$, the sums on the right approach the limit $\int_a^b [f(x) - g(x)] dx$ because f and g are continuous. The area of the region is defined to be the value of this integral. That is,

$$A = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n [f(c_k) - g(c_k)] \Delta x_k = \int_a^b [f(x) - g(x)] dx.$$

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves** $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f - g)$ from a to b :

$$A = \int_a^b [f(x) - g(x)] dx.$$

When applying this definition it is usually helpful to graph the curves. The graph reveals which curve is the upper curve f and which is the lower curve g . It also helps you find the limits of integration if they are not given. You may need to find where the curves intersect to determine the limits of integration, and this may involve solving the equation $f(x) = g(x)$ for values of x . Then you can integrate the function $f - g$ for the area between the intersections.

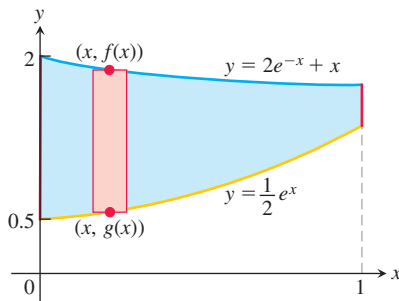


FIGURE 5.28 The region in Example 4 with a typical approximating rectangle.

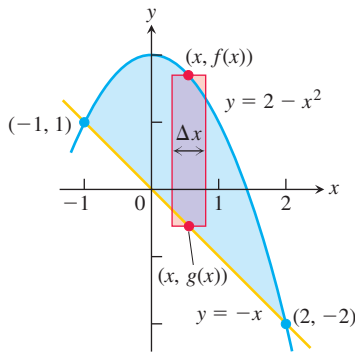


FIGURE 5.29 The region in Example 5 with a typical approximating rectangle from a Riemann sum.

EXAMPLE 4 Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

Solution Figure 5.28 displays the graphs of the curves and the region whose area we want to find. The area between the curves over the interval $0 \leq x \leq 1$ is

$$\begin{aligned} A &= \int_0^1 \left[(2e^{-x} + x) - \frac{1}{2}e^x \right] dx = \left[-2e^{-x} + \frac{1}{2}x^2 - \frac{1}{2}e^x \right]_0^1 \\ &= \left(-2e^{-1} + \frac{1}{2} - \frac{1}{2}e \right) - \left(-2 + 0 - \frac{1}{2} \right) \\ &= 3 - \frac{2}{e} - \frac{e}{2} \approx 0.9051. \end{aligned}$$

EXAMPLE 5 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves (Figure 5.29). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x .

$$\begin{aligned} 2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x). \\ x^2 - x - 2 &= 0 && \text{Rearrange terms.} \\ (x + 1)(x - 2) &= 0 && \text{Factor.} \\ x = -1, \quad x = 2. &&& \text{Solve.} \end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$. The area between the curves is

$$\begin{aligned} A &= \int_{-1}^2 [f(x) - g(x)] dx = \int_{-1}^2 [(2 - x^2) - (-x)] dx \\ &= \int_{-1}^2 (2 + x - x^2) dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2 \\ &= \left(4 + \frac{4}{2} - \frac{8}{3} \right) - \left(-2 + \frac{1}{2} - \frac{1}{3} \right) = \frac{9}{2}. \end{aligned}$$

If the formula for a bounding curve changes at one or more points, we subdivide the region into subregions that correspond to the formula changes and apply the formula for the area between curves to each subregion.

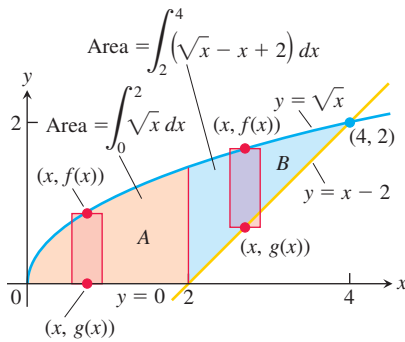


FIGURE 5.30 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 6.

EXAMPLE 6 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x -axis and the line $y = x - 2$.

Solution Figure 5.30 shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \leq x \leq 2$ to $g(x) = x - 2$ for $2 \leq x \leq 4$ (both formulas agree at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure 5.30.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x :

$$\sqrt{x} = x - 2 \quad \text{Equate } f(x) \text{ and } g(x).$$

$$x = (x - 2)^2 = x^2 - 4x + 4 \quad \text{Square both sides.}$$

$$x^2 - 5x + 4 = 0 \quad \text{Rewrite.}$$

$$(x - 1)(x - 4) = 0 \quad \text{Factor.}$$

$$x = 1, \quad x = 4. \quad \text{Solve.}$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

$$\text{For } 0 \leq x \leq 2: \quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$$

$$\text{For } 2 \leq x \leq 4: \quad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$$

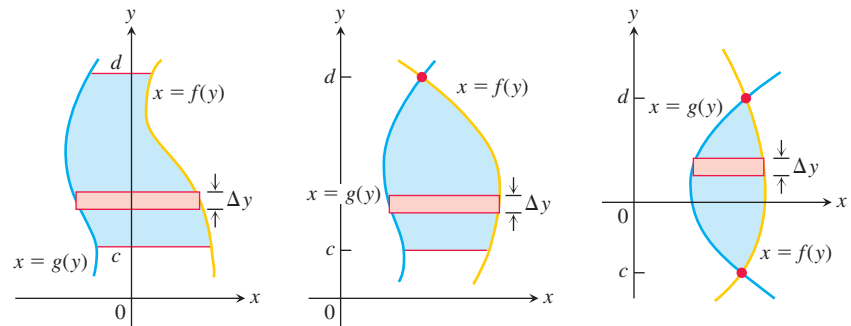
We add the areas of subregions A and B to find the total area:

$$\begin{aligned} \text{Total area} &= \underbrace{\int_0^2 \sqrt{x} \, dx}_{\text{area of A}} + \underbrace{\int_2^4 (\sqrt{x} - x + 2) \, dx}_{\text{area of B}} \\ &= \left[\frac{2}{3} x^{3/2} \right]_0^2 + \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x \right]_2^4 \\ &= \frac{2}{3} (2)^{3/2} - 0 + \left(\frac{2}{3} (4)^{3/2} - 8 + 8 \right) - \left(\frac{2}{3} (2)^{3/2} - 2 + 4 \right) \\ &= \frac{2}{3} (8) - 2 = \frac{10}{3}. \end{aligned}$$

Integration with Respect to y

If a region's bounding curves are described by functions of y , the approximating rectangles are horizontal instead of vertical, and the basic formula has y in place of x .

To find the areas of regions like these:



use the formula

$$A = \int_c^d [f(y) - g(y)] \, dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

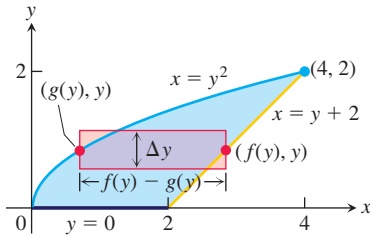


FIGURE 5.31 It takes two integrations to find the area of this region if we integrate with respect to x . It takes only one if we integrate with respect to y (Example 7).

EXAMPLE 7 Find the area of the region in Example 6 by integrating with respect to y .

Solution We first sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y -values (Figure 5.31). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y :

$$\begin{aligned} y + 2 &= y^2 && \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2. \\ y^2 - y - 2 &= 0 && \text{Rewrite.} \\ (y + 1)(y - 2) &= 0 && \text{Factor.} \\ y = -1, \quad y = 2. &&& \text{Solve.} \end{aligned}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x -axis.)

The area of the region is

$$\begin{aligned} A &= \int_c^d [f(y) - g(y)] dy = \int_0^2 [y + 2 - y^2] dy \\ &= \int_0^2 [2 + y - y^2] dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3} \right]_0^2 \\ &= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}. \end{aligned}$$

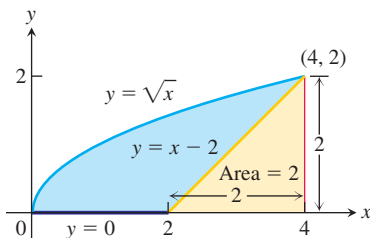


FIGURE 5.32 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

This is the result of Example 6, found with less work. ■

Although it was easier to find the area in Example 6 by integrating with respect to y rather than x (as we did in Example 7), there is an easier way yet. Looking at Figure 5.32, we see that the area we want is the area between the curve $y = \sqrt{x}$ and the x -axis for $0 \leq x \leq 4$, *minus* the area of an isosceles triangle of base and height equal to 2. So by combining calculus with some geometry, we find

$$\begin{aligned} \text{Area} &= \int_0^4 \sqrt{x} dx - \frac{1}{2}(2)(2) \\ &= \frac{2}{3}x^{3/2} \Big|_0^4 - 2 \\ &= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}. \end{aligned}$$

EXERCISES 5.6

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 7 to evaluate the integrals in Exercises 1–48.

1. a. $\int_0^3 \sqrt{y+1} dy$

b. $\int_{-1}^0 \sqrt{y+1} dy$

2. a. $\int_0^1 r\sqrt{1-r^2} dr$

b. $\int_{-1}^1 r\sqrt{1-r^2} dr$

3. a. $\int_0^{\pi/4} \tan x \sec^2 x dx$

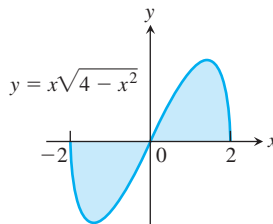
b. $\int_{-\pi/4}^0 \tan x \sec^2 x dx$

4. a. $\int_0^\pi 3 \cos^2 x \sin x \, dx$ b. $\int_{2\pi}^{3\pi} 3 \cos^2 x \sin x \, dx$ 29. $\int_1^2 \frac{2 \ln x}{x} \, dx$ 30. $\int_2^4 \frac{dx}{x \ln x}$
5. a. $\int_0^1 t^3 (1 + t^4)^3 \, dt$ b. $\int_{-1}^1 t^3 (1 + t^4)^3 \, dt$ 31. $\int_2^4 \frac{dx}{x(\ln x)^2}$ 32. $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$
6. a. $\int_0^{\sqrt{7}} t(t^2 + 1)^{1/3} \, dt$ b. $\int_{-\sqrt{7}}^0 t(t^2 + 1)^{1/3} \, dt$ 33. $\int_0^{\pi/2} \tan \frac{x}{2} \, dx$ 34. $\int_{\pi/4}^{\pi/2} \cot t \, dt$
7. a. $\int_{-1}^1 \frac{5r}{(4 + r^2)^2} \, dr$ b. $\int_0^1 \frac{5r}{(4 + r^2)^2} \, dr$ 35. $\int_0^{\pi/3} \tan^2 \theta \cos \theta \, d\theta$ 36. $\int_0^{\pi/12} 6 \tan 3x \, dx$
8. a. $\int_0^1 \frac{10\sqrt{v}}{(1 + v^{3/2})^2} \, dv$ b. $\int_1^4 \frac{10\sqrt{v}}{(1 + v^{3/2})^2} \, dv$ 37. $\int_{-\pi/2}^{\pi/2} \frac{2 \cos \theta \, d\theta}{1 + (\sin \theta)^2}$ 38. $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x \, dx}{1 + (\cot x)^2}$
9. a. $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2 + 1}} \, dx$ b. $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2 + 1}} \, dx$ 39. $\int_0^{\ln \sqrt{3}} \frac{e^x \, dx}{1 + e^{2x}}$ 40. $\int_1^{e^{\pi/4}} \frac{4 \, dt}{t(1 + \ln^2 t)}$
10. a. $\int_0^1 \frac{x^3}{\sqrt{x^4 + 9}} \, dx$ b. $\int_{-1}^0 \frac{x^3}{\sqrt{x^4 + 9}} \, dx$ 41. $\int_0^1 \frac{4 \, ds}{\sqrt{4 - s^2}}$ 42. $\int_0^{(3/4)\sqrt{2}} \frac{ds}{\sqrt{9 - 4s^2}}$
11. a. $\int_0^1 t\sqrt{4 + 5t} \, dt$ b. $\int_1^9 t\sqrt{4 + 5t} \, dt$ 43. $\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1} x) \, dx}{x\sqrt{x^2 - 1}}$ 44. $\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x) \, dx}{x\sqrt{x^2 - 1}}$
12. a. $\int_0^{\pi/6} (1 - \cos 3t) \sin 3t \, dt$ b. $\int_{\pi/6}^{\pi/3} (1 - \cos 3t) \sin 3t \, dt$ 45. $\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2 - 1}}$ 46. $\int_0^3 \frac{y \, dy}{\sqrt{5y + 1}}$
13. a. $\int_0^{2\pi} \frac{\cos z}{\sqrt{4 + 3 \sin z}} \, dz$ b. $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4 + 3 \sin z}} \, dz$ 47. $\int_0^1 \frac{\tan^{-1} x}{1 + x^2} \, dx$ 48. $\int_{-\sqrt{3}}^{1/\sqrt{3}} \frac{\cos(\tan^{-1} x)}{1 + 9x^2} \, dx$
14. a. $\int_{-\pi/2}^0 \left(2 + \tan \frac{t}{2}\right) \sec^2 \frac{t}{2} \, dt$ b. $\int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2}\right) \sec^2 \frac{t}{2} \, dt$
15. $\int_0^1 \sqrt{t^5 + 2t} (5t^4 + 2) \, dt$ 16. $\int_1^4 \frac{dy}{2\sqrt{y}(1 + \sqrt{y})^2}$
17. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta \, d\theta$ 18. $\int_{\pi}^{3\pi/2} \cot^5 \left(\frac{\theta}{6}\right) \sec^2 \left(\frac{\theta}{6}\right) \, d\theta$
19. $\int_0^{\pi} 5(5 - 4 \cos t)^{1/4} \sin t \, dt$ 20. $\int_0^{\pi/4} (1 - \sin 2t)^{3/2} \cos 2t \, dt$
21. $\int_0^1 (4y - y^2 + 4y^3 + 1)^{-2/3} (12y^2 - 2y + 4) \, dy$
22. $\int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2} (y^2 + 4y - 4) \, dy$
23. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta} \cos^2(\theta^{3/2}) \, d\theta$ 24. $\int_{-1}^{-1/2} t^{-2} \sin^2 \left(1 + \frac{1}{t}\right) \, dt$
25. $\int_0^{\pi/4} (1 + e^{\tan \theta}) \sec^2 \theta \, d\theta$ 26. $\int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta \, d\theta$
27. $\int_0^{\pi} \frac{\sin t}{2 - \cos t} \, dt$ 28. $\int_0^{\pi/3} \frac{4 \sin \theta}{1 - 4 \cos \theta} \, d\theta$

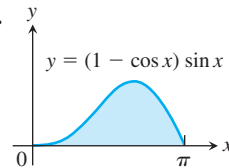
Area

Find the total areas of the shaded regions in Exercises 49–64.

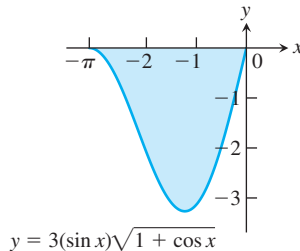
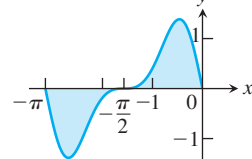
49.



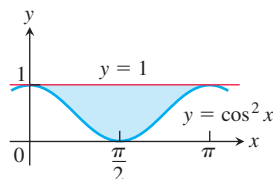
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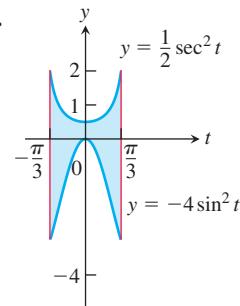
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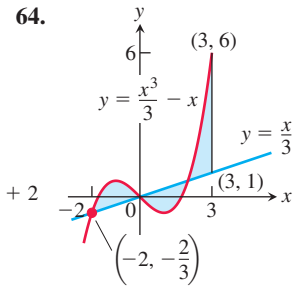
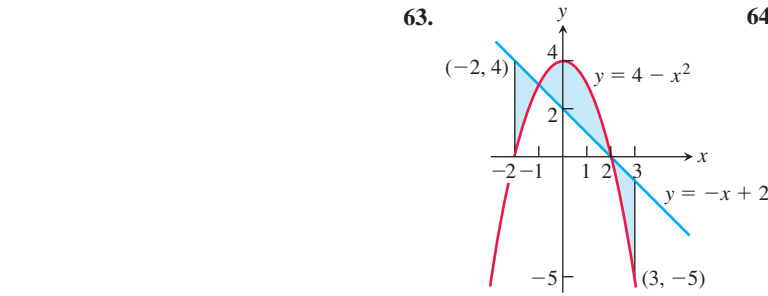
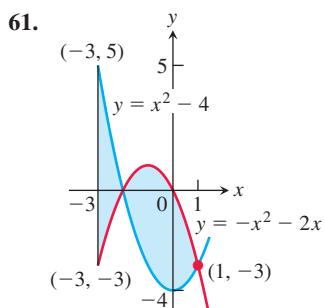
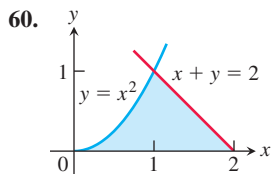
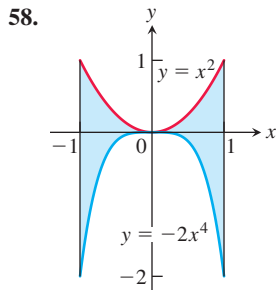
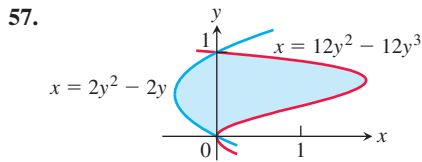
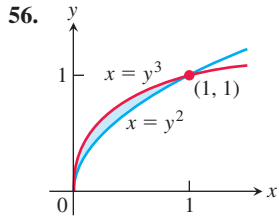
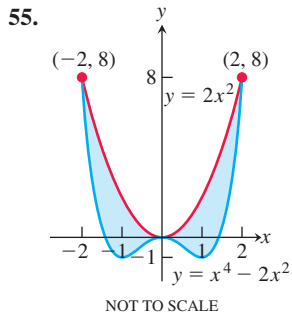
52. $y = \frac{\pi}{2} (\cos x)(\sin(\pi + \pi \sin x))$ 

53.



54.





Find the areas of the regions enclosed by the lines and curves in Exercises 65–74.

65. $y = x^2 - 2$ and $y = 2$

66. $y = 2x - x^2$ and $y = -3$

67. $y = x^4$ and $y = 8x$

68. $y = x^2 - 2x$ and $y = x$

69. $y = x^2$ and $y = -x^2 + 4x$

70. $y = 7 - 2x^2$ and $y = x^2 + 4$

71. $y = x^4 - 4x^2 + 4$ and $y = x^2$

72. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

73. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

74. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 75–82.

75. $x = 2y^2$, $x = 0$, and $y = 3$

76. $x = y^2$ and $x = y + 2$

77. $y^2 - 4x = 4$ and $4x - y = 16$

78. $x - y^2 = 0$ and $x + 2y^2 = 3$

79. $x + y^2 = 0$ and $x + 3y^2 = 2$

80. $x - y^{2/3} = 0$ and $x + y^4 = 2$

81. $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

82. $x = y^3 - y^2$ and $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 83–86.

83. $4x^2 + y = 4$ and $x^4 - y = 1$

84. $x^3 - y = 0$ and $3x^2 - y = 4$

85. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

86. $x + y^2 = 3$ and $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 87–94.

87. $y = 2 \sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

88. $y = 8 \cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

89. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

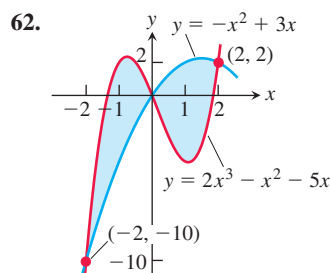
90. $y = \sin(\pi x/2)$ and $y = x$

91. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

92. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

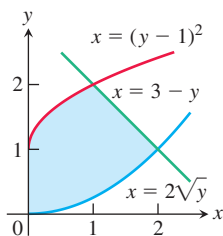
93. $x = 3 \sin y \sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

94. $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

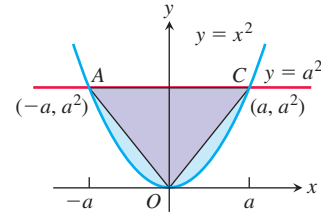


Area Between Curves

95. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.
96. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.
97. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x -axis.
98. Find the area of the “triangular” region in the first quadrant bounded on the left by the y -axis and on the right by the curves $y = \sin x$ and $y = \cos x$.
99. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.
100. Find the area between the curve $y = \tan x$ and the x -axis from $x = -\pi/4$ to $x = \pi/3$.
101. Find the area of the “triangular” region in the first quadrant that is bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.
102. Find the area of the “triangular” region in the first quadrant that is bounded above by the curve $y = e^{x/2}$, below by the curve $y = e^{-x/2}$, and on the right by the line $x = 2 \ln 2$.
103. Find the area of the region between the curve $y = 2x/(1 + x^2)$ and the interval $-2 \leq x \leq 2$ of the x -axis.
104. Find the area of the region between the curve $y = 2^{1-x}$ and the interval $-1 \leq x \leq 1$ of the x -axis.
105. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.
- Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c , what are the coordinates of the points where the line and parabola intersect? Add them to your figure.
 - Find c by integrating with respect to y . (This puts c in the limits of integration.)
 - Find c by integrating with respect to x . (This puts c into the integrand as well.)
106. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **a.** x , **b.** y .
107. Find the area of the region in the first quadrant bounded on the left by the y -axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.
108. Find the area of the region in the first quadrant bounded on the left by the y -axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.



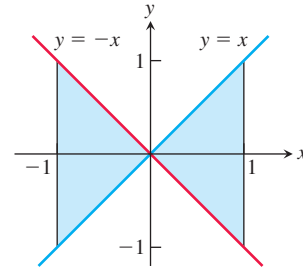
109. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.



110. Suppose the area of the region between the graph of a positive continuous function f and the x -axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.
111. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

a. $\int_{-1}^1 (x - (-x)) dx = \int_{-1}^1 2x dx$

b. $\int_{-1}^1 (-x - (x)) dx = \int_{-1}^1 -2x dx$



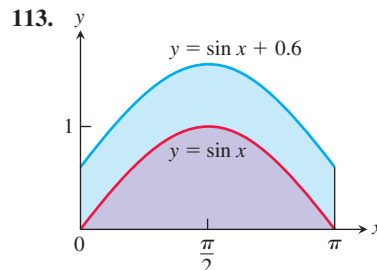
112. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ ($a < b$) is

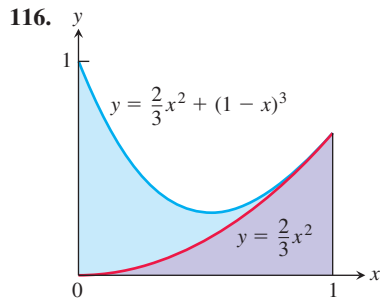
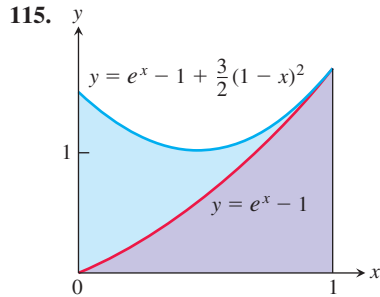
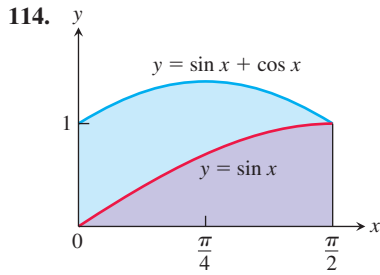
$$\int_a^b [f(x) - g(x)] dx.$$

Give reasons for your answer.

Comparing Areas

Compute the areas of the light blue and dark blue regions in each of Exercises 113 to 116 and determine which is larger, or show that they have equal area.





Theory and Examples

117. Suppose that $F(x)$ is an antiderivative of $f(x) = (\sin x)/x$, $x > 0$. Express

$$\int_1^3 \frac{\sin 2x}{x} dx$$

in terms of F .

118. Show that if f is continuous, then

$$\int_0^1 f(x) dx = \int_0^1 f(1-x) dx.$$

119. Suppose that

$$\int_0^1 f(x) dx = 3.$$

Find

$$\int_{-1}^0 f(x) dx$$

if **a.** f is odd, **b.** f is even.

120. **a.** Show that if f is odd on $[-a, a]$, then

$$\int_{-a}^a f(x) dx = 0.$$

b. Test the result in part (a) with $f(x) = \sin x$ and $a = \pi/2$.

121. If f is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x) dx}{f(x) + f(a-x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I .

122. By using a substitution, prove that for all positive numbers x and y ,

$$\int_x^{xy} \frac{1}{t} dt = \int_1^y \frac{1}{t} dt.$$

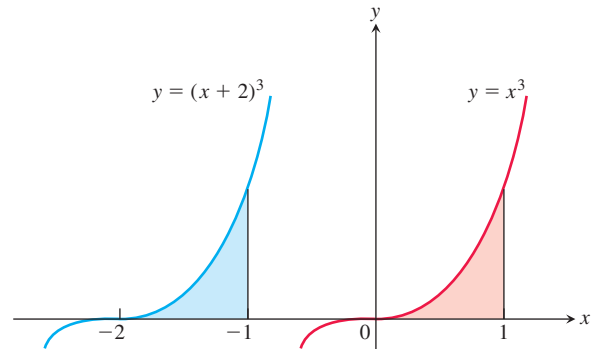
The Shift Property for Definite Integrals A basic property of definite integrals is their invariance under translation, as expressed by the equation

$$\int_a^b f(x) dx = \int_{a-c}^{b-c} f(x+c) dx \quad (1)$$

The equation holds whenever f is integrable and defined for the necessary values of x . For example, in the accompanying figure, show that

$$\int_{-2}^{-1} (x+2)^3 dx = \int_0^1 x^3 dx$$

because the areas of the shaded regions are congruent.



123. Use a substitution to verify Equation (1).

124. For each of the following functions, graph $f(x)$ over $[a, b]$ and $f(x+c)$ over $[a-c, b-c]$ to convince yourself that Equation (1) is reasonable.

- $f(x) = x^2$, $a = 0$, $b = 1$, $c = 1$
- $f(x) = \sin x$, $a = 0$, $b = \pi$, $c = \pi/2$
- $f(x) = \sqrt{x-4}$, $a = 4$, $b = 8$, $c = 5$

COMPUTER EXPLORATIONS

In Exercises 125–128, you will find the area between curves in the plane when you cannot find their points of intersection using simple algebra. Use a CAS to perform the following steps:

- Plot the curves together to see what they look like and how many points of intersection they have.
 - Use the numerical equation solver in your CAS to find all the points of intersection.
 - Integrate $|f(x) - g(x)|$ over consecutive pairs of intersection values.
 - Sum together the integrals found in part (c).
125. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$, $g(x) = x - 1$
126. $f(x) = \frac{x^4}{2} - 3x^3 + 10$, $g(x) = 8 - 12x$
127. $f(x) = x + \sin(2x)$, $g(x) = x^3$
128. $f(x) = x^2 \cos x$, $g(x) = x^3 - x$

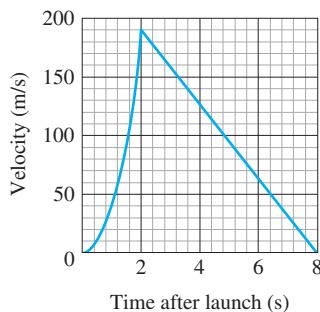
CHAPTER 5 Questions to Guide Your Review

- How can you sometimes estimate quantities like distance traveled, area, and average value with finite sums? Why might you want to do so?
- What is sigma notation? What advantage does it offer? Give examples.
- What is a Riemann sum? Why might you want to consider such a sum?
- What is the norm of a partition of a closed interval?
- What is the definite integral of a function f over a closed interval $[a, b]$? When can you be sure it exists?
- What is the relation between definite integrals and area? Describe some other interpretations of definite integrals.
- What is the average value of an integrable function over a closed interval? Must the function assume its average value? Explain.
- Describe the rules for working with definite integrals (Table 5.6). Give examples.
- What is the Fundamental Theorem of Calculus? Why is it so important? Illustrate each part of the theorem with an example.
- What is the Net Change Theorem? What does it say about the integral of velocity? The integral of marginal cost?
- Discuss how the processes of integration and differentiation can be considered as “inverses” of each other.
- How does the Fundamental Theorem provide a solution to the initial value problem $dy/dx = f(x)$, $y(x_0) = y_0$, when f is continuous?
- How is integration by substitution related to the Chain Rule?
- How can you sometimes evaluate indefinite integrals by substitution? Give examples.
- How does the method of substitution work for definite integrals? Give examples.
- How do you define and calculate the area of the region between the graphs of two continuous functions? Give an example.

CHAPTER 5 Practice Exercises

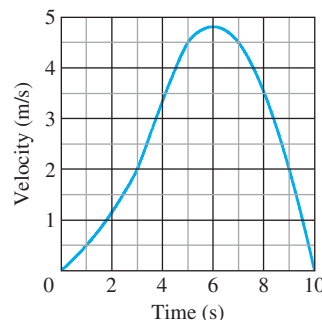
Finite Sums and Estimates

- The accompanying figure shows the graph of the velocity (m/s) of a model rocket for the first 8 s after launch. The rocket accelerated straight up for the first 2 s and then coasted to reach its maximum height at $t = 8$ s.



- Assuming that the rocket was launched from ground level, about how high did it go? (This is the rocket in Section 3.4, Exercise 17, but you do not need to do Exercise 17 to do the exercise here.)
 - Sketch a graph of the rocket's height above ground as a function of time for $0 \leq t \leq 8$.
- The accompanying figure shows the velocity (m/s) of a body moving along the s -axis during the time interval from $t = 0$ to $t = 10$ s. About how far did the body travel during those 10 s?

- Sketch a graph of s as a function of t for $0 \leq t \leq 10$, assuming $s(0) = 0$.



- Suppose that $\sum_{k=1}^{10} a_k = -2$ and $\sum_{k=1}^{10} b_k = 25$. Find the value of

- $\sum_{k=1}^{10} \frac{a_k}{4}$
- $\sum_{k=1}^{10} (b_k - 3a_k)$
- $\sum_{k=1}^{10} (a_k + b_k - 1)$
- $\sum_{k=1}^{10} \left(\frac{5}{2} - b_k \right)$

- Suppose that $\sum_{k=1}^{20} a_k = 0$ and $\sum_{k=1}^{20} b_k = 7$. Find the value of

- $\sum_{k=1}^{20} 3a_k$
- $\sum_{k=1}^{20} (a_k + b_k)$
- $\sum_{k=1}^{20} \left(\frac{1}{2} - \frac{2b_k}{7} \right)$
- $\sum_{k=1}^{20} (a_k - 2)$

Definite Integrals

In Exercises 5–8, express each limit as a definite integral. Then evaluate the integral to find the value of the limit. In each case, P is a partition of the given interval, and the numbers c_k are chosen from the subintervals of P .

5. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (2c_k - 1)^{-1/2} \Delta x_k$, where P is a partition of $[1, 5]$
6. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n c_k (c_k^2 - 1)^{1/3} \Delta x_k$, where P is a partition of $[1, 3]$
7. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \left(\cos\left(\frac{c_k}{2}\right) \right) \Delta x_k$, where P is a partition of $[-\pi, 0]$
8. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (\sin c_k)(\cos c_k) \Delta x_k$, where P is a partition of $[0, \pi/2]$
9. If $\int_{-2}^2 3f(x) dx = 12$, $\int_{-2}^5 f(x) dx = 6$, and $\int_{-2}^5 g(x) dx = 2$, find the value of each of the following.
 - a. $\int_{-2}^2 f(x) dx$
 - b. $\int_{-2}^5 f(x) dx$
 - c. $\int_5^{-2} g(x) dx$
 - d. $\int_{-2}^5 (-\pi g(x)) dx$
 - e. $\int_{-2}^5 \left(\frac{f(x) + g(x)}{5} \right) dx$
10. If $\int_0^2 f(x) dx = \pi$, $\int_0^2 7g(x) dx = 7$, and $\int_0^1 g(x) dx = 2$, find the value of each of the following.
 - a. $\int_0^2 g(x) dx$
 - b. $\int_1^2 g(x) dx$
 - c. $\int_2^0 f(x) dx$
 - d. $\int_0^2 \sqrt{2} f(x) dx$
 - e. $\int_0^2 (g(x) - 3f(x)) dx$

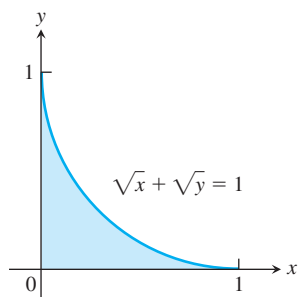
Area

In Exercises 11–14, find the total area of the region between the graph of f and the x -axis.

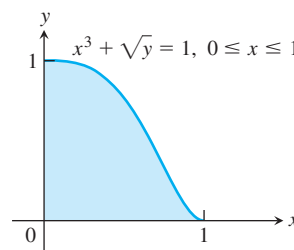
11. $f(x) = x^2 - 4x + 3$, $0 \leq x \leq 3$
12. $f(x) = 1 - (x^2/4)$, $-2 \leq x \leq 3$
13. $f(x) = 5 - 5x^{2/3}$, $-1 \leq x \leq 8$
14. $f(x) = 1 - \sqrt{x}$, $0 \leq x \leq 4$

Find the areas of the regions enclosed by the curves and lines in Exercises 15–26.

15. $y = x$, $y = 1/x^2$, $x = 2$
16. $y = x$, $y = 1/\sqrt{x}$, $x = 2$
17. $\sqrt{x} + \sqrt{y} = 1$, $x = 0$, $y = 0$



18. $x^3 + \sqrt{y} = 1$, $x = 0$, $y = 0$, for $0 \leq x \leq 1$



19. $x = 2y^2$, $x = 0$, $y = 3$
20. $x = 4 - y^2$, $x = 0$
21. $y^2 = 4x$, $y = 4x - 2$
22. $y^2 = 4x + 4$, $y = 4x - 16$
23. $y = \sin x$, $y = x$, $0 \leq x \leq \pi/4$
24. $y = |\sin x|$, $y = 1$, $-\pi/2 \leq x \leq \pi/2$
25. $y = 2 \sin x$, $y = \sin 2x$, $0 \leq x \leq \pi$
26. $y = 8 \cos x$, $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$
27. Find the area of the “triangular” region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$.
28. Find the area of the “triangular” region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 1$.
29. Find the extreme values of $f(x) = x^3 - 3x^2$, and find the area of the region enclosed by the graph of f and the x -axis.
30. Find the area of the region cut from the first quadrant by the curve $x^{1/2} + y^{1/2} = a^{1/2}$.
31. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and the lines $x = y$ and $y = -1$.
32. Find the total area of the region between the curves $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq 3\pi/2$.
33. Find the area between the curve $y = 2(\ln x)/x$ and the x -axis from $x = 1$ to $x = e$.
34. a. Show that the area between the curve $y = 1/x$ and the x -axis from $x = 10$ to $x = 20$ is the same as the area between the curve and the x -axis from $x = 1$ to $x = 2$.
b. Show that the area between the curve $y = 1/x$ and the x -axis from ka to kb is the same as the area between the curve and the x -axis from $x = a$ to $x = b$ ($0 < a < b$, $k > 0$).

Initial Value Problems

35. Show that $y = x^2 + \int_1^x \frac{1}{t} dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}; \quad y'(1) = 3, \quad y(1) = 1.$$

36. Show that $y = \int_0^x (1 + 2\sqrt{\sec t}) dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = \sqrt{\sec x} \tan x; \quad y'(0) = 3, \quad y(0) = 0.$$

Express the solutions of the initial value problems in Exercises 37 and 38 in terms of integrals.

$$37. \frac{dy}{dx} = \frac{\sin x}{x}, \quad y(5) = -3$$

$$38. \frac{dy}{dx} = \sqrt{2 - \sin^2 x}, \quad y(-1) = 2$$

Solve the initial value problems in Exercises 39–42.

39. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad y(0) = 0$

40. $\frac{dy}{dx} = \frac{1}{x^2 + 1} - 1, \quad y(0) = 1$

41. $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}, \quad x > 1; \quad y(2) = \pi$

42. $\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}}, \quad y(0) = 2$

For Exercises 43 and 44, find a function f that satisfies each equation.

43. $f(x) = 1 + \int_1^x t f(t) dt$ 44. $f(x) = \int_0^x (1 + f(t)^2) dt$

Evaluating Indefinite Integrals

Evaluate the integrals in Exercises 45–76.

45. $\int 2(\cos x)^{-1/2} \sin x dx$ 46. $\int (\tan x)^{-3/2} \sec^2 x dx$

47. $\int (2\theta + 1 + 2 \cos(2\theta + 1)) d\theta$

48. $\int \left(\frac{1}{\sqrt{2\theta - \pi}} + 2 \sec^2(2\theta - \pi) \right) d\theta$

49. $\int \left(t - \frac{2}{t} \right) \left(t + \frac{2}{t} \right) dt$

50. $\int \frac{(t+1)^2 - 1}{t^4} dt$ 51. $\int \sqrt{t} \sin(2t^{3/2}) dt$

52. $\int (\sec \theta \tan \theta) \sqrt{1 + \sec \theta} d\theta$

53. $\int e^x \sec^2(e^x - 7) dx$

54. $\int e^y \csc(e^y + 1) \cot(e^y + 1) dy$

55. $\int (\sec^2 x) e^{\tan x} dx$ 56. $\int (\csc^2 x) e^{\cot x} dx$

57. $\int_{-1}^1 \frac{dx}{3x - 4}$ 58. $\int_1^e \frac{\sqrt{\ln x}}{x} dx$

59. $\int_0^4 \frac{2t}{t^2 - 25} dt$ 60. $\int \frac{\tan(\ln v)}{v} dv$

61. $\int \frac{(\ln x)^{-3}}{x} dx$ 62. $\int \frac{1}{r} \csc^2(1 + \ln r) dr$

63. $\int x 3^{x^2} dx$ 64. $\int 2^{\tan x} \sec^2 x dx$

65. $\int \frac{3 dr}{\sqrt{1 - 4(r-1)^2}}$ 66. $\int \frac{6 dr}{\sqrt{4 - (r+1)^2}}$

67. $\int \frac{dx}{2 + (x-1)^2}$ 68. $\int \frac{dx}{1 + (3x+1)^2}$

69. $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2 - 4}}$ 70. $\int \frac{dx}{(x+3)\sqrt{(x+3)^2 - 25}}$

71. $\int \frac{e^{\arcsin \sqrt{x}} dx}{2\sqrt{x-x^2}}$ 72. $\int \frac{\sqrt{\arcsin x} dx}{\sqrt{1-x^2}}$

73. $\int \frac{dy}{\sqrt{\arctan y} (1+y^2)}$ 74. $\int \frac{(\arctan x)^2 dx}{1+x^2}$

75. $\int \frac{\sin 2\theta - \cos 2\theta}{(\sin 2\theta + \cos 2\theta)^3} d\theta$ 76. $\int \cos \theta \cdot \sin(\sin \theta) d\theta$

Evaluating Definite Integrals

Evaluate the integrals in Exercises 77–116.

77. $\int_{-1}^1 (3x^2 - 4x + 7) dx$ 78. $\int_0^1 (8s^3 - 12s^2 + 5) ds$

79. $\int_1^2 \frac{4}{v^2} dv$ 80. $\int_1^{27} x^{-4/3} dx$

81. $\int_1^4 \frac{dt}{t\sqrt{t}}$ 82. $\int_1^4 \frac{(1 + \sqrt{u})^{1/2}}{\sqrt{u}} du$

83. $\int_0^1 \frac{36 dx}{(2x+1)^3}$ 84. $\int_0^1 \frac{dr}{\sqrt[3]{(7-5r)^2}}$

85. $\int_{1/8}^1 x^{-1/3} (1 - x^{2/3})^{3/2} dx$ 86. $\int_0^{1/2} x^3 (1 + 9x^4)^{-3/2} dx$

87. $\int_0^\pi \sin^2 5r dr$ 88. $\int_0^{\pi/4} \cos^2 \left(4t - \frac{\pi}{4} \right) dt$

89. $\int_0^{\pi/3} \sec^2 \theta d\theta$ 90. $\int_{\pi/4}^{3\pi/4} \csc^2 x dx$

91. $\int_\pi^{3\pi} \cot^2 \frac{x}{6} dx$ 92. $\int_0^\pi \tan^2 \frac{\theta}{3} d\theta$

93. $\int_{-\pi/3}^0 \sec x \tan x dx$ 94. $\int_{\pi/4}^{3\pi/4} \csc z \cot z dz$

95. $\int_0^{\pi/2} 5(\sin x)^{3/2} \cos x dx$ 96. $\int_{-\pi/2}^{\pi/2} 15 \sin^4 3x \cos 3x dx$

97. $\int_0^{\pi/2} \frac{3 \sin x \cos x}{\sqrt{1 + 3 \sin^2 x}} dx$ 98. $\int_0^{\pi/4} \frac{\sec^2 x}{(1 + 7 \tan x)^{2/3}} dx$

99. $\int_1^4 \left(\frac{x}{8} + \frac{1}{2x} \right) dx$ 100. $\int_1^8 \left(\frac{2}{3x} - \frac{8}{x^2} \right) dx$

101. $\int_{-2}^{-1} e^{-(x+1)} dx$ 102. $\int_{-\ln 2}^0 e^{2w} dw$

103. $\int_0^{\ln 5} e^r (3e^r + 1)^{-3/2} dr$ 104. $\int_0^{\ln 9} e^\theta (e^\theta - 1)^{1/2} d\theta$

105. $\int_1^e \frac{1}{x} (1 + 7 \ln x)^{-1/3} dx$ 106. $\int_1^3 \frac{(\ln(v+1))^2}{v+1} dv$

107. $\int_1^8 \frac{\log_4 \theta}{\theta} d\theta$ 108. $\int_1^e \frac{8 \ln 3 \log_3 \theta}{\theta} d\theta$

109. $\int_{-3/4}^{3/4} \frac{6 dx}{\sqrt{9 - 4x^2}}$ 110. $\int_{-1/5}^{1/5} \frac{6 dx}{\sqrt{4 - 25x^2}}$

111. $\int_{-2}^2 \frac{3 dt}{4 + 3t^2}$ 112. $\int_{\sqrt{3}}^3 \frac{dt}{3 + t^2}$

113. $\int_{1/\sqrt{3}}^1 \frac{dy}{y\sqrt{4y^2 - 1}}$ 114. $\int_{4\sqrt{2}}^8 \frac{24 dy}{y\sqrt{y^2 - 16}}$

115. $\int_{\sqrt{2}/3}^{2/3} \frac{dy}{|y|\sqrt{9y^2 - 1}}$ 116. $\int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{dy}{|y|\sqrt{5y^2 - 3}}$

Average Values

117. Find the average value of $f(x) = mx + b$

a. over $[-1, 1]$

b. over $[-k, k]$

118. Find the average value of

a. $y = \sqrt{3x}$ over $[0, 3]$

b. $y = \sqrt{ax}$ over $[0, a]$

119. Let f be a function that is differentiable on $[a, b]$. In Chapter 2 we defined the average rate of change of f over $[a, b]$ to be

$$\frac{f(b) - f(a)}{b - a}$$

and the instantaneous rate of change of f at x to be $f'(x)$. In this chapter we defined the average value of a function. For the new definition of average to be consistent with the old one, we should have

$$\frac{f(b) - f(a)}{b - a} = \text{average value of } f' \text{ on } [a, b].$$

Is this the case? Give reasons for your answer.

120. Is it true that the average value of an integrable function over an interval of length 2 is half the function's integral over the interval? Give reasons for your answer.
121. a. Verify that $\int \ln x \, dx = x \ln x - x + C$.
b. Find the average value of $\ln x$ over $[1, e]$.
122. Find the average value of $f(x) = 1/x$ on $[1, 2]$.

- T** 123. Compute the average value of the temperature function

$$f(x) = 20 \sin\left(\frac{2\pi}{365}(x - 101)\right) - 4$$

for a 365-day year. (See Exercise 98, Section 3.6.) This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is -3.5°C , which is slightly higher than the average value of $f(x)$.

- T** 124. **Specific heat of a gas** Specific heat C_v is the amount of heat required to raise the temperature of one mole (gram molecule) of a gas with constant volume by 1°C . The specific heat of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Find the average value of C_v for $20^\circ\text{C} \leq T \leq 675^\circ\text{C}$ and the temperature at which it is attained.

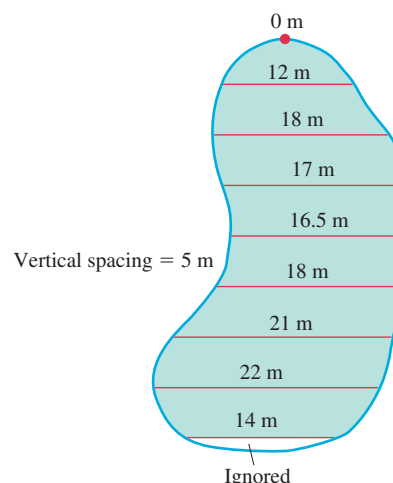
Differentiating Integrals

In Exercises 125–132, find dy/dx .

125. $y = \int_2^x \sqrt{2 + \cos^3 t} \, dt$ 126. $y = \int_2^{7x^2} \sqrt{2 + \cos^3 t} \, dt$
127. $y = \int_x^1 \frac{6}{3 + t^4} \, dt$ 128. $y = \int_{\sec x}^2 \frac{1}{t^2 + 1} \, dt$
129. $y = \int_{\ln x^2}^0 e^{\cos t} \, dt$ 130. $y = \int_1^{e^{\sqrt{x}}} \ln(t^2 + 1) \, dt$
131. $y = \int_0^{\sin^{-1} x} \frac{dt}{\sqrt{1 - 2t^2}}$ 132. $y = \int_{\tan^{-1} x}^{\pi/4} e^{\sqrt{t}} \, dt$

Theory and Examples

133. Is it true that every function $y = f(x)$ that is differentiable on $[a, b]$ is itself the derivative of some function on $[a, b]$? Give reasons for your answer.
134. Suppose that $f(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_0^1 \sqrt{1 + x^4} \, dx$ in terms of F and give a reason for your answer.
135. Find dy/dx if $y = \int_x^1 \sqrt{1 + t^2} \, dt$. Explain the main steps in your calculation.
136. Find dy/dx if $y = \int_{\cos x}^0 (1/(1 - t^2)) \, dt$. Explain the main steps in your calculation.
137. **A new parking lot** To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for \$10,000. The cost to clear the land will be \$1.00 a square meter, and the lot will cost \$2.00 a square meter to pave. Can the job be done for \$10,000? Use a lower sum estimate to see. (Answers may vary slightly, depending on the estimate used.)



138. Skydivers A and B are in a helicopter hovering at 2000 m. Skydiver A jumps and descends for 4 s before opening her parachute. The helicopter then climbs to 2200 m and hovers there. Forty-five seconds after A leaves the aircraft, B jumps and descends for 13 s before opening his parachute. Both skydivers descend at 4.9 m/s with parachutes open. Assume that the skydivers fall freely (no effective air resistance) before their parachutes open.
- At what altitude does A's parachute open?
 - At what altitude does B's parachute open?
 - Which skydiver lands first?

CHAPTER 5 Additional and Advanced Exercises

Theory and Examples

1. a. If $\int_0^1 7f(x) \, dx = 7$, does $\int_0^1 f(x) \, dx = 1$?
b. If $\int_0^1 f(x) \, dx = 4$ and $f(x) \geq 0$, does

$$\int_0^1 \sqrt{f(x)} \, dx = \sqrt{4} = 2?$$

Give reasons for your answers.

2. Suppose $\int_{-2}^2 f(x) \, dx = 4$, $\int_2^5 f(x) \, dx = 3$, $\int_{-2}^5 g(x) \, dx = 2$. Which, if any, of the following statements are true?

- a. $\int_5^2 f(x) \, dx = -3$ b. $\int_{-2}^5 (f(x) + g(x)) \, dx = 9$
c. $f(x) \leq g(x)$ on the interval $-2 \leq x \leq 5$

3. Initial value problem Show that

$$y = \frac{1}{a} \int_0^x f(t) \sin a(x-t) dt$$

solves the initial value problem

$$\frac{d^2y}{dx^2} + a^2y = f(x), \quad \frac{dy}{dx} = 0 \text{ and } y = 0 \text{ when } x = 0.$$

(Hint: $\sin(ax - at) = \sin ax \cos at - \cos ax \sin at$.)**4. Proportionality** Suppose that x and y are related by the equation

$$x = \int_0^y \frac{1}{\sqrt{1+4t^2}} dt.$$

Show that d^2y/dx^2 is proportional to y , and find the constant of proportionality.**5.** Find $f(4)$ if

$$\text{a. } \int_0^{x^2} f(t) dt = x \cos \pi x \quad \text{b. } \int_0^{f(x)} t^2 dt = x \cos \pi x.$$

6. Find $f(\pi/2)$ from the following information.i) f is positive and continuous.ii) The area under the curve $y = f(x)$ from $x = 0$ to $x = a$ is

$$\frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a.$$

7. The area of the region in the xy -plane enclosed by the x -axis, the curve $y = f(x)$, $f(x) \geq 0$, and the lines $x = 1$ and $x = b$ is equal to $\sqrt{b^2 + 1} - \sqrt{2}$ for all $b > 1$. Find $f(x)$.

8. Prove that

$$\int_0^x \left(\int_0^u f(t) dt \right) du = \int_0^x f(u) (x-u) du.$$

(Hint: Express the integral on the right-hand side as the difference of two integrals. Then show that both sides of the equation have the same derivative with respect to x .)

9. Finding a curve Find the equation for the curve in the xy -plane that passes through the point $(1, -1)$ if its slope at x is always $3x^2 + 2$.

10. Shoveling dirt You sling a shovelful of dirt up from the bottom of a hole with an initial velocity of 9.8 m/s. The dirt must rise 5.2 m above the release point to clear the edge of the hole. Is that enough speed to get the dirt out, or had you better duck?

Piecewise Continuous Functions

Although we are mainly interested in continuous functions, many functions in applications are piecewise continuous. A function $f(x)$ is **piecewise continuous on a closed interval I** if f has only finitely many discontinuities in I , the limits

$$\lim_{x \rightarrow c^-} f(x) \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x)$$

exist and are finite at every interior point of I , and the appropriate one-sided limits exist and are finite at the endpoints of I . All piecewise continuous functions are integrable. The points of discontinuity subdivide I into open and half-open subintervals on which f is continuous, and the limit criteria above guarantee that f has a continuous extension to the closure of each subinterval. To integrate a piecewise continuous

function, we integrate the individual extensions and add the results. The integral of

$$f(x) = \begin{cases} 1-x, & -1 \leq x < 0 \\ x^2, & 0 \leq x < 2 \\ -1, & 2 \leq x \leq 3 \end{cases}$$

(Figure 5.33) over $[-1, 3]$ is

$$\begin{aligned} \int_{-1}^3 f(x) dx &= \int_{-1}^0 (1-x) dx + \int_0^2 x^2 dx + \int_2^3 (-1) dx \\ &= \left[x - \frac{x^2}{2} \right]_{-1}^0 + \left[\frac{x^3}{3} \right]_0^2 + \left[-x \right]_2^3 \\ &= \frac{3}{2} + \frac{8}{3} - 1 = \frac{19}{6}. \end{aligned}$$

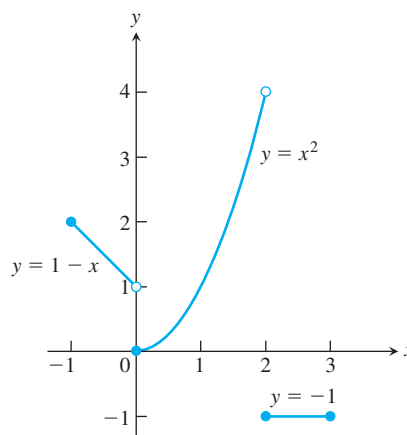


FIGURE 5.33 Piecewise continuous functions like this are integrated piece by piece.

The Fundamental Theorem applies to piecewise continuous functions with the restriction that $(d/dx) \int_a^x f(t) dt$ is expected to equal $f(x)$ only at values of x at which f is continuous. There is a similar restriction on Leibniz's Rule (see Exercises 25–32).

Graph the functions in Exercises 11–16 and integrate them over their domains.

$$\text{11. } f(x) = \begin{cases} x^{2/3}, & -8 \leq x < 0 \\ -4, & 0 \leq x \leq 3 \end{cases}$$

$$\text{12. } f(x) = \begin{cases} \sqrt{-x}, & -4 \leq x < 0 \\ x^2 - 4, & 0 \leq x \leq 3 \end{cases}$$

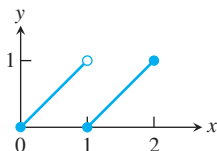
$$\text{13. } g(t) = \begin{cases} t, & 0 \leq t < 1 \\ \sin \pi t, & 1 \leq t \leq 2 \end{cases}$$

$$\text{14. } h(z) = \begin{cases} \sqrt{1-z}, & 0 \leq z < 1 \\ (7z-6)^{-1/3}, & 1 \leq z \leq 2 \end{cases}$$

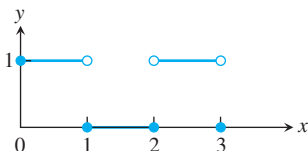
$$\text{15. } f(x) = \begin{cases} 1, & -2 \leq x < -1 \\ 1-x^2, & -1 \leq x < 1 \\ 2, & 1 \leq x \leq 2 \end{cases}$$

$$\text{16. } h(r) = \begin{cases} r, & -1 \leq r < 0 \\ 1-r^2, & 0 \leq r < 1 \\ 1, & 1 \leq r \leq 2 \end{cases}$$

17. Find the average value of the function graphed in the accompanying figure.



18. Find the average value of the function graphed in the accompanying figure.



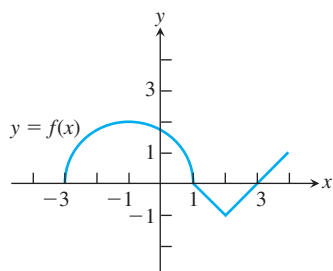
Limits

Find the limits in Exercises 19–22.

19. $\lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$ 20. $\lim_{x \rightarrow \infty} \frac{1}{x} \int_0^x \arctan t \, dt$
21. $\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$
22. $\lim_{n \rightarrow \infty} \frac{1}{n} (e^{1/n} + e^{2/n} + \cdots + e^{(n-1)/n} + e^{n/n})$

Defining Functions Using the Fundamental Theorem

23. **A function defined by an integral** The graph of a function f consists of a semicircle and two line segments as shown. Let $g(x) = \int_1^x f(t) \, dt$.



- a. Find $g(1)$. b. Find $g(3)$. c. Find $g(-1)$.
- d. Find all values of x on the open interval $(-3, 4)$ at which g has a relative maximum.
- e. Write an equation for the line tangent to the graph of g at $x = -1$.
- f. Find the x -coordinate of each point of inflection of the graph of g on the open interval $(-3, 4)$.
- g. Find the range of g .
24. **A differential equation** Show that both of the following conditions are satisfied by $y = \sin x + \int_x^\pi \cos 2t \, dt + 1$:
- i) $y'' = -\sin x + 2 \sin 2x$
- ii) $y = 1$ and $y' = -2$ when $x = \pi$.

Leibniz's Rule In applications, we sometimes encounter functions defined by integrals that have variable upper limits of integration and variable lower limits of integration at the same time. We can find the derivative of such an integral by a formula called **Leibniz's Rule**.

Leibniz's Rule

If f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t) \, dt = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}.$$

To prove the rule, let F be an antiderivative of f on $[a, b]$. Then

$$\int_{u(x)}^{v(x)} f(t) \, dt = F(v(x)) - F(u(x)).$$

Differentiating both sides of this equation with respect to x gives the equation we want:

$$\begin{aligned} \frac{d}{dx} \int_{u(x)}^{v(x)} f(t) \, dt &= \frac{d}{dx} [F(v(x)) - F(u(x))] \\ &= F'(v(x)) \frac{dv}{dx} - F'(u(x)) \frac{du}{dx} \quad \text{Chain Rule} \\ &= f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}. \end{aligned}$$

Use Leibniz's Rule to find the derivatives of the functions in Exercises 25–32.

25. $f(x) = \int_{1/x}^x \frac{1}{t} \, dt$ 26. $f(x) = \int_{\cos x}^{\sin x} \frac{1}{1-t^2} \, dt$
27. $g(y) = \int_{\sqrt{y}}^{2\sqrt{y}} \sin t^2 \, dt$ 28. $g(y) = \int_{\sqrt{y}}^{y^2} \frac{e^t}{t} \, dt$
29. $y = \int_{x^2/2}^{x^2} \ln \sqrt{t} \, dt$ 30. $y = \int_{\sqrt{x}}^{3\sqrt{x}} \ln t \, dt$
31. $y = \int_0^{\ln x} \sin e^t \, dt$ 32. $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt$

Theory and Examples

33. Use Leibniz's Rule to find the value of x that maximizes the value of the integral

$$\int_x^{x+3} t(5-t) \, dt.$$

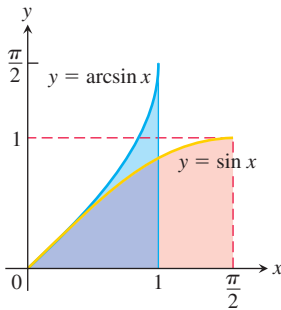
34. For what $x > 0$ does $x^{(x^x)} = (x^x)^x$? Give reasons for your answer.
35. For the two curves $y = 2(\log_2 x)/x$ and $y = 2(\log_4 x)/x$:
- a. Find the area between the first curve and the x -axis from $x = 1$ to $x = e$.
- b. Find the area between the second curve and the x -axis from $x = 1$ to $x = e$.
- c. Consider the two areas obtained in parts (a) and (b). What is the ratio of the larger area to the smaller?
36. a. Find df/dx if

$$f(x) = \int_1^{e^x} \frac{2 \ln t}{t} \, dt.$$

- b. Find $f(0)$.
- c. What can you conclude about the graph of f ? Give reasons for your answer.
37. Find $f'(2)$ if $f(x) = e^{g(x)}$ and $g(x) = \int_2^x \frac{t}{1+t^4} \, dt$.

38. Use the accompanying figure to show that

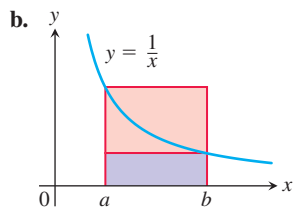
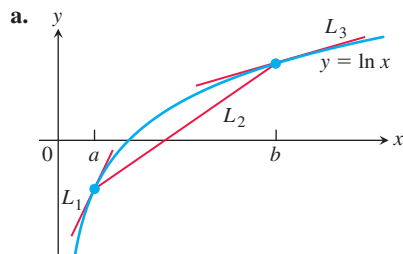
$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{2} - \int_0^1 \arcsin x \, dx.$$



39. **Napier's inequality** Here are two pictorial proofs that

$$b > a > 0 \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}.$$

Explain what is going on in each case.



(Source: Roger B. Nelson, *College Mathematics Journal*, Vol. 24, No. 2, March 1993, p. 165.)

40. **Bound on an integral** Let f be a continuously differentiable function on $[a, b]$ satisfying $\int_a^b f(x) \, dx = 0$.

a. If $c = (a + b)/2$, show that

$$\int_a^b x f(x) \, dx = \int_a^c (x - c) f(x) \, dx + \int_c^b (x - c) f(x) \, dx.$$

b. Let $t = |x - c|$ and $\ell = (b - a)/2$. Show that

$$\int_a^b x f(x) \, dx = \int_0^\ell t(f(c + t) - f(c - t)) \, dt.$$

c. Apply the Mean Value Theorem from Section 4.2 to part (b) to prove that

$$\left| \int_a^b x f(x) \, dx \right| \leq \frac{(b - a)^3}{12} M,$$

where M is the absolute maximum of f' on $[a, b]$.

Approximating Finite Sums with Integrals

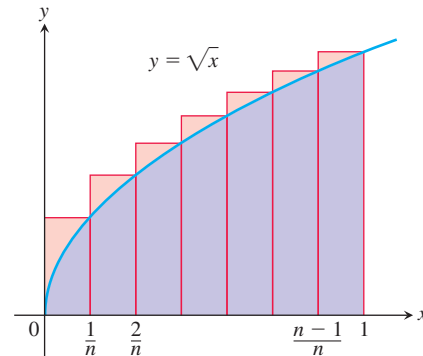
In many applications of calculus, integrals are used to approximate finite sums—the reverse of the usual procedure of using finite sums to approximate integrals.

For example, let's estimate the sum of the square roots of the first n positive integers, $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$. The integral

$$\int_0^1 \sqrt{x} \, dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}$$

is the limit of the upper sums

$$\begin{aligned} S_n &= \sqrt{\frac{1}{n}} \cdot \frac{1}{n} + \sqrt{\frac{2}{n}} \cdot \frac{1}{n} + \cdots + \sqrt{\frac{n}{n}} \cdot \frac{1}{n} \\ &= \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}}. \end{aligned}$$



Therefore, when n is large, S_n will be close to $2/3$ and we will have

$$\text{Root sum} = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} = S_n \cdot n^{3/2} \approx \frac{2}{3} n^{3/2}.$$

The following table shows how good the approximation can be.

n	Root sum	$(2/3)n^{3/2}$	Relative error
10	22.468	21.082	$1.386/22.468 \approx 6\%$
50	239.04	235.70	1.4%
100	671.46	666.67	0.7%
1000	21,097	21,082	0.07%

41. Evaluate

$$\lim_{n \rightarrow \infty} \frac{1^5 + 2^5 + 3^5 + \cdots + n^5}{n^6}$$

by showing that the limit is

$$\int_0^1 x^5 \, dx$$

and evaluating the integral.

42. See Exercise 41. Evaluate

$$\lim_{n \rightarrow \infty} \frac{1}{n^4} (1^3 + 2^3 + 3^3 + \cdots + n^3).$$

43. Let $f(x)$ be a continuous function. Express

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right) \right]$$

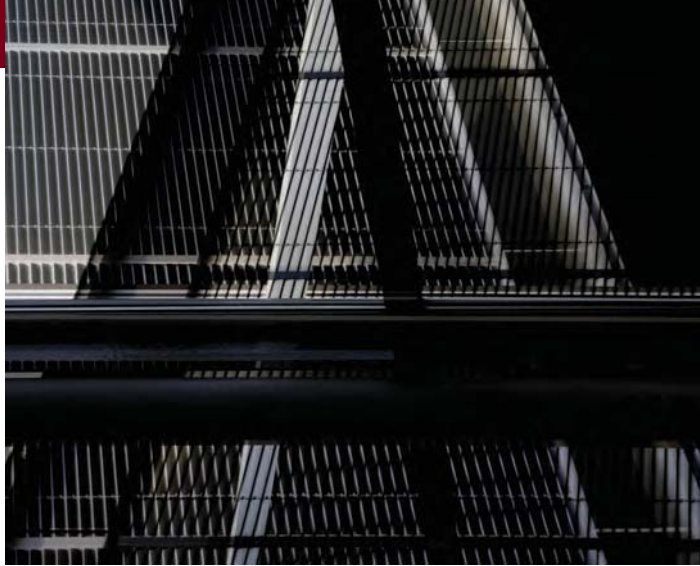
as a definite integral.

44. Use the result of Exercise 43 to evaluate

a. $\lim_{n \rightarrow \infty} \frac{1}{n^2} (2 + 4 + 6 + \cdots + 2n),$

8

Techniques of Integration



OVERVIEW The Fundamental Theorem tells us how to evaluate a definite integral once we have an antiderivative for the integrand function. However, finding antiderivatives (or indefinite integrals) is not as straightforward as finding derivatives. In this chapter we study a number of important techniques that apply to finding integrals for specialized classes of functions such as trigonometric functions, products of certain functions, and rational functions. Since we cannot always find an antiderivative, we develop numerical methods for calculating definite integrals. We also study integrals for which the domain or range is infinite, called *improper integrals*.

8.1 Using Basic Integration Formulas

Table 8.1 summarizes the indefinite integrals of many of the functions we have studied so far, and the substitution method helps us use the table to evaluate more complicated functions involving these basic ones. In this section we combine the Substitution Rules (studied in Chapter 5) with algebraic methods and trigonometric identities to help us use Table 8.1. A more extensive Table of Integrals is given at the back of the chapter, and we discuss its use in Section 8.6.

Sometimes we have to rewrite an integral to match it to a standard form of the type displayed in Table 8.1. We start with an example of this procedure.

EXAMPLE 1 Evaluate the integral

$$\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx.$$

Solution We rewrite the integral and apply the Substitution Rule for Definite Integrals presented in Section 5.6, to find

$$\begin{aligned} \int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx &= \int_1^{11} \frac{du}{\sqrt{u}} \quad \begin{array}{l} u = x^2 - 3x + 1, du = (2x - 3) dx; \\ u = 1 \text{ when } x = 3, u = 11 \text{ when } x = 5 \end{array} \\ &= \int_1^{11} u^{-1/2} du \\ &= 2\sqrt{u} \Big|_1^{11} = 2(\sqrt{11} - 1) \approx 4.63. \quad \text{Table 8.1, Formula 2} \end{aligned}$$

TABLE 8.1 Basic integration formulas

1. $\int k \, dx = kx + C$ (any number k)	12. $\int \tan x \, dx = \ln \sec x + C$
2. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ ($n \neq -1$)	13. $\int \cot x \, dx = \ln \sin x + C$
3. $\int \frac{dx}{x} = \ln x + C$	14. $\int \sec x \, dx = \ln \sec x + \tan x + C$
4. $\int e^x \, dx = e^x + C$	15. $\int \csc x \, dx = -\ln \csc x + \cot x + C$
5. $\int a^x \, dx = \frac{a^x}{\ln a} + C$ ($a > 0, a \neq 1$)	16. $\int \sinh x \, dx = \cosh x + C$
6. $\int \sin x \, dx = -\cos x + C$	17. $\int \cosh x \, dx = \sinh x + C$
7. $\int \cos x \, dx = \sin x + C$	18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
8. $\int \sec^2 x \, dx = \tan x + C$	19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
9. $\int \csc^2 x \, dx = -\cot x + C$	20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left \frac{x}{a}\right + C$
10. $\int \sec x \tan x \, dx = \sec x + C$	21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$ ($a > 0$)
11. $\int \csc x \cot x \, dx = -\csc x + C$	22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$ ($x > a > 0$)

EXAMPLE 2 Complete the square to evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\ &= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\ &= \int \frac{du}{\sqrt{a^2 - u^2}} && a = 4, u = (x - 4), \\ &&& du = dx \\ &= \arcsin\left(\frac{u}{a}\right) + C && \text{Table 8.1, Formula 18} \\ &= \arcsin\left(\frac{x - 4}{4}\right) + C. \end{aligned}$$

EXAMPLE 3 Evaluate the integral

$$\int (\cos x \sin 2x + \sin x \cos 2x) dx.$$

Solution We can replace the integrand with an equivalent trigonometric expression using the Sine Addition Formula to obtain a simple substitution:

$$\begin{aligned} \int (\cos x \sin 2x + \sin x \cos 2x) dx &= \int \sin(x + 2x) dx \\ &= \int \sin 3x dx \\ &= \int \frac{1}{3} \sin u du \quad u = 3x, du = 3 dx \\ &= -\frac{1}{3} \cos 3x + C. \quad \text{Table 8.1, Formula 6} \end{aligned}$$

In Section 5.5 we found the indefinite integral of the secant function by multiplying it by a fractional form equal to one, and then integrating the equivalent result. We can use that same procedure in other instances as well, as we illustrate next.

EXAMPLE 4 Find $\int_0^{\pi/4} \frac{dx}{1 - \sin x}$.

Solution We multiply the numerator and denominator of the integrand by $1 + \sin x$. This procedure transforms the integral into one we can evaluate:

$$\begin{aligned} \int_0^{\pi/4} \frac{dx}{1 - \sin x} &= \int_0^{\pi/4} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} dx && \text{Multiply and divide by conjugate.} \\ &= \int_0^{\pi/4} \frac{1 + \sin x}{1 - \sin^2 x} dx && \text{Simplify.} \\ &= \int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x} dx && 1 - \sin^2 x = \cos^2 x \\ &= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x) dx && \text{Use Table 8.1, Formulas 8 and 10} \\ &= \left[\tan x + \sec x \right]_0^{\pi/4} = (1 + \sqrt{2} - (0 + 1)) = \sqrt{2}. \end{aligned}$$

EXAMPLE 5 Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2} dx.$$

Solution The integrand is an improper fraction since the degree of the numerator is greater than the degree of the denominator. To integrate it, we perform long division to obtain a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

$$\begin{array}{r} x - 3 \\ 3x + 2 \overline{) 3x^2 - 7x} \\ \underline{3x^2 + 2x} \\ -9x \\ \underline{-9x - 6} \\ +6 \end{array}$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2} dx = \int \left(x - 3 + \frac{6}{3x + 2} \right) dx = \frac{x^2}{2} - 3x + 2 \ln|3x + 2| + C. \quad \blacksquare$$

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 8.5.

EXAMPLE 6 Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx = 3 \int \frac{x dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \quad du = -2x dx, \quad \text{so} \quad x dx = -\frac{1}{2} du.$$

Then we obtain

$$\begin{aligned} 3 \int \frac{x dx}{\sqrt{1 - x^2}} &= 3 \int \frac{(-1/2) du}{\sqrt{u}} = -\frac{3}{2} \int u^{-1/2} du \\ &= -\frac{3}{2} \cdot \frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1. \end{aligned}$$

The second of the new integrals is a standard form,

$$2 \int \frac{dx}{\sqrt{1 - x^2}} = 2 \arcsin x + C_2. \quad \text{Table 8.1, Formula 18}$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx = -3\sqrt{1 - x^2} + 2 \arcsin x + C. \quad \blacksquare$$

The question of what to substitute for in an integrand is not always quite so clear. Sometimes we simply proceed by trial-and-error, and if nothing works out, we then try another method altogether. The next several sections of the text present some of these new methods, but substitution works in the following example.

EXAMPLE 7 Evaluate

$$\int \frac{dx}{(1 + \sqrt{x})^3}.$$

Solution We might try substituting for the term $\sqrt{x}$, but the derivative factor $1/\sqrt{x}$ is missing from the integrand, so this substitution will not help. The other possibility is to substitute for $(1 + \sqrt{x})$, and it turns out this works:

$$\begin{aligned} \int \frac{dx}{(1 + \sqrt{x})^3} &= \int \frac{2(u - 1) du}{u^3} & u = 1 + \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx; \\ & & dx = 2\sqrt{x} du = 2(u - 1) du \\ &= \int \left(\frac{2}{u^2} - \frac{2}{u^3} \right) du \end{aligned}$$

$$\begin{aligned}
&= -\frac{2}{u} + \frac{1}{u^2} + C \\
&= \frac{1-2u}{u^2} + C \\
&= \frac{1-2(1+\sqrt{x})}{(1+\sqrt{x})^2} + C \\
&= C - \frac{1+2\sqrt{x}}{(1+\sqrt{x})^2}.
\end{aligned}$$

When evaluating definite integrals, a property of the integrand may help us in calculating the result.

EXAMPLE 8 Evaluate

$$\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx.$$

Solution No substitution or algebraic manipulation is clearly helpful here. But we observe that the interval of integration is the symmetric interval $[-\pi/2, \pi/2]$. Moreover, the factor x^3 is an odd function, and $\cos x$ is an even function, so their product is odd. Therefore,

$$\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx = 0. \quad \text{Theorem 8, Section 5.6}$$

EXERCISES 8.1

Assorted Integrations

The integrals in Exercises 1–44 are in no particular order. Evaluate each integral using any algebraic method, trigonometric identity, or substitution you think is appropriate.

1. $\int_0^1 \frac{16x}{8x^2 + 2} \, dx$
2. $\int \frac{x^2}{x^2 + 1} \, dx$
3. $\int (\sec x - \tan x)^2 \, dx$
4. $\int_{\pi/4}^{\pi/3} \frac{dx}{\cos^2 x \tan x}$
5. $\int \frac{1-x}{\sqrt{1-x^2}} \, dx$
6. $\int \frac{dx}{x - \sqrt{x}}$
7. $\int \frac{e^{-\cot z}}{\sin^2 z} \, dz$
8. $\int \frac{2 \ln z^3}{16z} \, dz$
9. $\int \frac{dz}{e^z + e^{-z}}$
10. $\int_1^2 \frac{8 \, dx}{x^2 - 2x + 2}$
11. $\int_{-1}^0 \frac{4 \, dx}{1 + (2x + 1)^2}$
12. $\int_{-1}^3 \frac{4x^2 - 7}{2x + 3} \, dx$
13. $\int \frac{dt}{1 - \sec t}$
14. $\int \csc t \sin 3t \, dt$
15. $\int_0^{\pi/4} \frac{1 + \sin \theta}{\cos^2 \theta} \, d\theta$
16. $\int \frac{d\theta}{\sqrt{2\theta - \theta^2}}$
17. $\int \frac{\ln y}{y + 4y \ln^2 y} \, dy$
18. $\int \frac{2\sqrt{y}}{2\sqrt{y}} \, dy$
19. $\int \frac{d\theta}{\sec \theta + \tan \theta}$
20. $\int \frac{dt}{t\sqrt{3} + t^2}$
21. $\int \frac{4t^3 - t^2 + 16t}{t^2 + 4} \, dt$
22. $\int \frac{x + 2\sqrt{x-1}}{2x\sqrt{x-1}} \, dx$
23. $\int_0^{\pi/2} \sqrt{1 - \cos \theta} \, d\theta$
24. $\int (\sec t + \cot t)^2 \, dt$
25. $\int \frac{dy}{\sqrt{e^{2y} - 1}}$
26. $\int \frac{6 \, dy}{\sqrt{y}(1+y)}$
27. $\int \frac{2 \, dx}{x\sqrt{1 - 4 \ln^2 x}}$
28. $\int \frac{dx}{(x-2)\sqrt{x^2 - 4x + 3}}$
29. $\int (\csc x - \sec x)(\sin x + \cos x) \, dx$
30. $\int 3 \sinh\left(\frac{x}{2} + \ln 5\right) \, dx$
31. $\int_{\sqrt{2}}^3 \frac{2x^3}{x^2 - 1} \, dx$
32. $\int_{-1}^1 \sqrt{1+x^2} \sin x \, dx$
33. $\int_{-1}^0 \sqrt{\frac{1+y}{1-y}} \, dy$
34. $\int e^{z+e^z} \, dz$

35. $\int \frac{7 dx}{(x-1)\sqrt{x^2-2x-48}}$ 36. $\int \frac{dx}{(2x+1)\sqrt{4x+4x^2}}$
37. $\int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5} d\theta$ 38. $\int \frac{d\theta}{\cos \theta - 1}$
39. $\int \frac{dx}{1+e^x}$ 40. $\int \frac{\sqrt{x}}{1+x^3} dx$
Hint: Use long division. *Hint: Let $u = x^{3/2}$.*
41. $\int \frac{e^{3x}}{e^x + 1} dx$ 42. $\int \frac{2^x - 1}{3^x} dx$
43. $\int \frac{1}{\sqrt{x}(1+x)} dx$ 44. $\int \frac{\tan \theta + 3}{\sin \theta} d\theta$

Theory and Examples

45. **Area** Find the area of the region bounded above by $y = 2 \cos x$ and below by $y = \sec x$, $-\pi/4 \leq x \leq \pi/4$.
46. **Volume** Find the volume of the solid generated by revolving the region in Exercise 45 about the x -axis.
47. **Arc length** Find the length of the curve $y = \ln(\cos x)$, $0 \leq x \leq \pi/3$.
48. **Arc length** Find the length of the curve $y = \ln(\sec x)$, $0 \leq x \leq \pi/4$.
49. **Centroid** Find the centroid of the region bounded by the x -axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.
50. **Centroid** Find the centroid of the region bounded by the x -axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

51. The functions $y = e^{x^3}$ and $y = x^3 e^{x^3}$ do not have elementary antiderivatives, but $y = (1 + 3x^3)e^{x^3}$ does. Evaluate

$$\int (1 + 3x^3) e^{x^3} dx.$$

52. Use the substitution $u = \tan x$ to evaluate the integral

$$\int \frac{dx}{1 + \sin^2 x}.$$

53. Use the substitution $u = x^4 + 1$ to evaluate the integral

$$\int x^7 \sqrt{x^4 + 1} dx.$$

54. **Using different substitutions** Show that the integral

$$\int ((x^2 - 1)(x + 1))^{-2/3} dx$$

can be evaluated with any of the following substitutions.

- a. $u = 1/(x + 1)$
- b. $u = ((x - 1)/(x + 1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1
- c. $u = \arctan x$
- d. $u = \tan^{-1} \sqrt{x}$
- e. $u = \tan^{-1}((x - 1)/2)$
- f. $u = \arccos x$
- g. $u = \cosh^{-1} x$

What is the value of the integral?

8.2 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int u(x)v'(x) dx.$$

It is useful when u can be differentiated repeatedly and v' can be integrated repeatedly without difficulty. The integrals

$$\int x \cos x dx \quad \text{and} \quad \int x^2 e^x dx$$

are such integrals because $u(x) = x$ or $u(x) = x^2$ can be differentiated repeatedly, and $v'(x) = \cos x$ or $v'(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int \ln x dx \quad \text{and} \quad \int e^x \cos x dx.$$

In the first case, the integrand $\ln x$ can be rewritten as $(\ln x)(1)$, and $u(x) = \ln x$ is easy to differentiate while $v'(x) = 1$ easily integrates to x . In the second case, each part of the integrand appears again after repeated differentiation or integration.

Product Rule in Integral Form

If u and v are differentiable functions of x , the Product Rule says that

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[u(x)v(x)] dx = \int [u'(x)v(x) + u(x)v'(x)] dx$$

or

$$\int \frac{d}{dx}[u(x)v(x)] dx = \int u'(x)v(x) dx + \int u(x)v'(x) dx.$$

Rearranging the terms of this last equation, we get

$$\int u(x)v'(x) dx = \int \frac{d}{dx}[u(x)v(x)] dx - \int v(x)u'(x) dx,$$

leading to the following **integration by parts** formula.

Integration by Parts Formula

$$\int u(x)v'(x) dx = u(x)v(x) - \int v(x)u'(x) dx \quad (1)$$

This formula allows us to exchange the problem of computing the integral $\int u(x)v'(x) dx$ for the problem of computing a different integral, $\int v(x)u'(x) dx$. In many cases, we can choose the functions u and v so that the second integral is easier to compute than the first. There can be many choices for u and v , and it is not always clear which choice works best, so sometimes we need to try several.

The formula is often given in differential form. With $v'(x) dx = dv$ and $u'(x) dx = du$, the integration by parts formula becomes

Integration by Parts Formula—Differential Version

$$\int u dv = uv - \int v du \quad (2)$$

The next examples illustrate the technique.

EXAMPLE 1 Find

$$\int x \cos x dx.$$

Solution There is no obvious antiderivative of $x \cos x$, so we use the integration by parts formula

$$\int u(x)v'(x) dx = u(x)v(x) - \int v(x)u'(x) dx$$

to change this expression to one that is easier to integrate. We first decide how to choose the functions $u(x)$ and $v(x)$. There is more than one way to do this, but here we choose to factor the expression $x \cos x$ into

$$u(x) = x \quad \text{and} \quad v'(x) = \cos x.$$

Next we differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = 1 \quad \text{and} \quad v(x) = \sin x.$$

When finding an antiderivative for $v'(x)$, we have a choice of how to pick a constant of integration C . We choose the constant $C = 0$, since that makes this antiderivative as simple as possible. We now apply the integration by parts formula:

$$\begin{aligned} \int \underbrace{x}_{u(x)} \underbrace{\cos x}_{v'(x)} dx &= \underbrace{x}_{u(x)} \underbrace{\sin x}_{v(x)} - \int \underbrace{\sin x}_{v(x)} \underbrace{(1)}_{u'(x)} dx && \text{Integration by parts formula} \\ &= x \sin x + \cos x + C && \text{Integrate and simplify.} \end{aligned}$$

and we have found the integral of the original function. ■

There are at least four apparent choices available for $u(x)$ and $v'(x)$ in Example 1:

1. Let $u(x) = 1$ and $v'(x) = x \cos x$.
2. Let $u(x) = x$ and $v'(x) = \cos x$.
3. Let $u(x) = x \cos x$ and $v'(x) = 1$.
4. Let $u(x) = \cos x$ and $v'(x) = x$.

We used choice 2 in Example 1. The other three choices lead to integrals that we do not know how to evaluate. For instance, Choice 3, with $u'(x) = \cos x - x \sin x$, leads to the integral

$$\int (x \cos x - x^2 \sin x) dx.$$

The goal of integration by parts is to go from an integral $\int u(x)v'(x) dx$ that we don't see how to evaluate to an integral $\int v(x)u'(x) dx$ that we can evaluate. Generally, we choose $v'(x)$ first to be as much of the integrand as we can readily integrate; then we let $u(x)$ be the leftover part. When finding $v(x)$ from $v'(x)$, any antiderivative will work, and we usually pick the simplest one. In particular, no arbitrary constant of integration is needed in $v(x)$ because it would simply cancel out of the right-hand side of Equation (2).

EXAMPLE 2 Find $\int \ln x dx$.

Solution We have not yet seen how to find an antiderivative for $\ln x$. If we set $u(x) = \ln x$, then $u'(x)$ is the simpler function $1/x$. It may not appear that a second function $v'(x)$ is multiplying $u(x) = \ln x$, but we can choose $v'(x)$ to be the constant function $v'(x) = 1$. We use the integration by parts formula given in Equation (1), with

$$u(x) = \ln x \quad \text{and} \quad v'(x) = 1.$$

We differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = \frac{1}{x} \quad \text{and} \quad v(x) = x.$$

Then

$$\begin{aligned} \int \underbrace{\ln x}_{u(x)} \cdot \underbrace{1}_{v'(x)} dx &= \underbrace{(\ln x)}_{u(x)} \underbrace{x}_{v(x)} - \int \underbrace{x}_{v(x)} \underbrace{\frac{1}{x}}_{u'(x)} dx && \text{Integration by parts formula} \\ &= x \ln x - \int 1 dx \\ &= x \ln x - x + C && \text{Simplify and integrate.} \end{aligned}$$

In the following examples we use the differential form to indicate the process of integration by parts. The computations are the same, with du and dv providing shorter expressions for $u'(x) dx$ and $v'(x) dx$.

Sometimes we have to use integration by parts more than once, as in the next example.

EXAMPLE 3 Evaluate

$$\int x^2 e^x dx.$$

Solution We use the integration by parts formula given in Equation (1), with

$$u(x) = x^2 \quad \text{and} \quad v'(x) = e^x.$$

We differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = 2x \quad \text{and} \quad v(x) = e^x.$$

We summarize this choice by setting $du = u'(x) dx$ and $dv = v'(x) dx$, so

$$du = 2x dx \quad \text{and} \quad dv = e^x dx.$$

We then have

$$\int \underbrace{x^2}_{u} \underbrace{e^x dx}_{dv} = \underbrace{x^2}_{u} \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{2x dx}_{du} = x^2 e^x - 2 \int x e^x dx \quad \text{Integration by parts formula}$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x dx$. Then $du = dx$, $v = e^x$, and

$$\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = \underbrace{x}_{u} \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{dx}_{du} = x e^x - e^x + C. \quad \begin{array}{l} \text{Integration by parts Equation (2)} \\ u = x, \quad dv = e^x dx \\ v = e^x, \quad du = dx \end{array}$$

Using this last evaluation, we then obtain

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\ &= x^2 e^x - 2x e^x + 2e^x + C, \end{aligned}$$

where the constant of integration is renamed after substituting for the integral on the right. ■

The technique of Example 3 works for any integral $\int x^n e^x dx$ in which n is a positive integer, because differentiating x^n will eventually lead to a constant, and repeatedly integrating e^x is easy.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 4 Evaluate

$$\int e^x \cos x dx.$$

Solution Let $u = e^x$ and $dv = \cos x dx$. Then $du = e^x dx$, $v = \sin x$, and

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \quad dv = \sin x dx, \quad v = -\cos x, \quad du = e^x dx.$$

Then

$$\begin{aligned} \int e^x \cos x dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x dx. \end{aligned}$$

The unknown integral now appears on both sides of the equation, but with opposite signs. Adding the integral to both sides and adding the constant of integration gives

$$2 \int e^x \cos x dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration then gives

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \quad \blacksquare$$

EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x \, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \quad \text{and} \quad dv = \cos x \, dx,$$

so that

$$du = (n-1)(\cos^{n-2} x)(-\sin x \, dx) \quad \text{and} \quad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned} \int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx. \end{aligned}$$

If we add

$$(n-1) \int \cos^n x \, dx$$

to both sides of this equation, we obtain

$$n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx.$$

We then divide through by n , and the final result is

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx. \quad \blacksquare$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\begin{aligned} \int \cos^3 x \, dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3} \int \cos x \, dx \\ &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C. \end{aligned}$$

Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both u' and v' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a^b u(x) v'(x) \, dx = u(x) v(x) \Big|_a^b - \int_a^b v(x) u'(x) \, dx \quad (3)$$

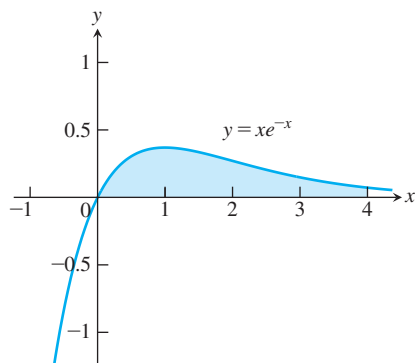


FIGURE 8.1 The region in Example 6.

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x -axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x} dx.$$

Let $u = x$, $dv = e^{-x} dx$, $v = -e^{-x}$, and $du = dx$. Then

$$\begin{aligned} \int_0^4 xe^{-x} dx &= -xe^{-x} \Big|_0^4 - \int_0^4 (-e^{-x}) dx && \text{Integration by parts Formula (3)} \\ &= [-4e^{-4} - (-0e^{-0})] + \int_0^4 e^{-x} dx \\ &= -4e^{-4} - e^{-x} \Big|_0^4 \\ &= -4e^{-4} - (e^{-4} - e^{-0}) = 1 - 5e^{-4} \approx 0.91. \end{aligned}$$

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2} dx$
2. $\int \theta \cos \pi \theta d\theta$
3. $\int t^2 \cos t dt$
4. $\int x^2 \sin x dx$
5. $\int_1^2 x \ln x dx$
6. $\int_1^e x^3 \ln x dx$
7. $\int x e^x dx$
8. $\int x e^{3x} dx$
9. $\int x^2 e^{-x} dx$
10. $\int (x^2 - 2x + 1) e^{2x} dx$
11. $\int \tan^{-1} y dy$
12. $\int \arcsin y dy$
13. $\int x \sec^2 x dx$
14. $\int 4x \sec^2 2x dx$
15. $\int x^3 e^x dx$
16. $\int p^4 e^{-p} dp$
17. $\int (x^2 - 5x) e^x dx$
18. $\int (r^2 + r + 1) e^r dr$
19. $\int x^5 e^x dx$
20. $\int t^2 e^{4t} dt$
21. $\int e^\theta \sin \theta d\theta$
22. $\int e^{-y} \cos y dy$
23. $\int e^{2x} \cos 3x dx$
24. $\int e^{-2x} \sin 2x dx$

Using Substitution

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}} ds$
26. $\int_0^1 x \sqrt{1-x} dx$
27. $\int_0^{\pi/3} x \tan^2 x dx$
28. $\int \ln(x + x^2) dx$

$$29. \int \sin(\ln x) dx$$

$$30. \int z(\ln z)^2 dz$$

Evaluating Integrals

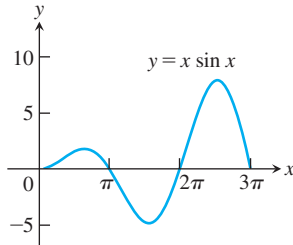
Evaluate the integrals in Exercises 31–56. Some integrals do not require integration by parts.

31. $\int x \sec x^2 dx$
32. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$
33. $\int x (\ln x)^2 dx$
34. $\int \frac{1}{x(\ln x)^2} dx$
35. $\int \frac{\ln x}{x^2} dx$
36. $\int \frac{(\ln x)^3}{x} dx$
37. $\int x^3 e^{x^4} dx$
38. $\int x^5 e^{x^3} dx$
39. $\int x^3 \sqrt{x^2 + 1} dx$
40. $\int x^2 \sin x^3 dx$
41. $\int \sin 3x \cos 2x dx$
42. $\int \sin 2x \cos 4x dx$
43. $\int \sqrt{x} \ln x dx$
44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$
45. $\int \cos \sqrt{x} dx$
46. $\int \sqrt{x} e^{\sqrt{x}} dx$
47. $\int_0^{\pi/2} \theta^2 \sin 2\theta d\theta$
48. $\int_0^{\pi/2} x^3 \cos 2x dx$
49. $\int_{2/\sqrt{3}}^2 t \sec^{-1} t dt$
50. $\int_0^{1/\sqrt{2}} 2x \arcsin(x^2) dx$
51. $\int x \arctan x dx$
52. $\int x^2 \tan^{-1} \frac{x}{2} dx$
53. $\int (1 + 2x^2) e^{x^2} dx$
54. $\int \frac{x e^x}{(x+1)^2} dx$
55. $\int \sqrt{x} (\arcsin \sqrt{x}) dx$
56. $\int \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}} dx$

Theory and Examples

57. Finding area Find the area of the region enclosed by the curve $y = x \sin x$ and the x -axis (see the accompanying figure) for

- $0 \leq x \leq \pi$.
- $\pi \leq x \leq 2\pi$.
- $2\pi \leq x \leq 3\pi$.
- What pattern do you see here? What is the area between the curve and the x -axis for $n\pi \leq x \leq (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.

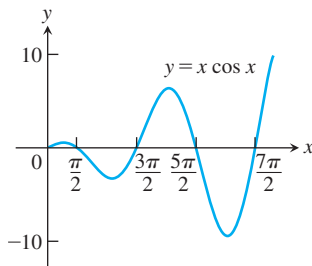


58. Finding area Find the area of the region enclosed by the curve $y = x \cos x$ and the x -axis (see the accompanying figure) for

- $\pi/2 \leq x \leq 3\pi/2$.
- $3\pi/2 \leq x \leq 5\pi/2$.
- $5\pi/2 \leq x \leq 7\pi/2$.
- What pattern do you see? What is the area between the curve and the x -axis for

$$\left(\frac{2n-1}{2}\right)\pi \leq x \leq \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



59. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

60. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

- about the y -axis.
- about the line $x = 1$.

61. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \leq x \leq \pi/2$, about

- the y -axis.
- the line $x = \pi/2$.

62. Finding volume Find the volume of the solid generated by revolving the region bounded by the x -axis and the curve $y = x \sin x$, $0 \leq x \leq \pi$, about

- the y -axis.
- the line $x = \pi$.

(See Exercise 57 for a graph.)

63. Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.

- Find the area of the region.
- Find the volume of the solid formed by revolving this region about the x -axis.
- Find the volume of the solid formed by revolving this region about the line $x = -2$.
- Find the centroid of the region.

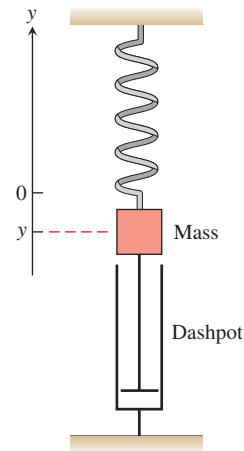
64. Consider the region bounded by the graphs of $y = \arctan x$, $y = 0$, and $x = 1$.

- Find the area of the region.
- Find the volume of the solid formed by revolving this region about the y -axis.

65. Average value A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \quad t \geq 0.$$

Find the average value of y over the interval $0 \leq t \leq 2\pi$.



66. Average value In a mass-spring-dashpot system like the one in Exercise 65, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \quad t \geq 0.$$

Find the average value of y over the interval $0 \leq t \leq 2\pi$.

Reduction Formulas

In Exercises 67–73, use integration by parts to establish the reduction formula.

$$67. \int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$$

$$68. \int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

$$69. \int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx, \quad a \neq 0$$

$$70. \int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$$

$$71. \int x^m (\ln x)^n dx = \frac{x^{m+1}}{m+1} (\ln x)^n - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx, \quad m \neq -1$$

$$72. \int x^n \sqrt{x+1} dx = \frac{2x^n}{2n+3} (x+1)^{3/2} - \frac{2n}{2n+3} \int x^{n-1} \sqrt{x+1} dx$$

$$73. \int \frac{x^n}{\sqrt{x+1}} dx = \frac{2x^n}{2n+1} \sqrt{x+1} - \frac{2n}{2n+1} \int \frac{x^{n-1}}{\sqrt{x+1}} dx$$

74. Use Example 5 to show that

$$\begin{aligned} \int_0^{\pi/2} \sin^n x dx &= \int_0^{\pi/2} \cos^n x dx \\ &= \begin{cases} \left(\frac{\pi}{2}\right) \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n}, & n \text{ even} \\ \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots n}, & n \text{ odd} \end{cases} \end{aligned}$$

75. Show that

$$\int_a^b \left(\int_x^b f(t) dt \right) dx = \int_a^b (x-a) f(x) dx.$$

76. Use integration by parts to obtain the formula

$$\int \sqrt{1-x^2} dx = \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx.$$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned} \int f^{-1}(x) dx &= \int y f'(y) dy && \begin{array}{l} y = f^{-1}(x), \quad x = f(y) \\ dx = f'(y) dy \end{array} \\ &= y f(y) - \int f(y) dy && \begin{array}{l} \text{Integration by parts with} \\ u = y, \quad dv = f'(y) dy \end{array} \\ &= x f^{-1}(x) - \int f(y) dy \end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned} \int \ln x dx &= \int y e^y dy && \begin{array}{l} y = \ln x, \quad x = e^y \\ dx = e^y dy \end{array} \\ &= y e^y - e^y + C \\ &= x \ln x - x + C. \end{aligned}$$

For the integral of $\arccos x$, we get

$$\begin{aligned} \int \arccos x dx &= x \arccos x - \int \cos y dy && y = \arccos x \\ &= x \arccos x - \sin y + C \\ &= x \arccos x - \sin(\arccos x) + C. \end{aligned}$$

Use the formula

$$\int f^{-1}(x) dx = x f^{-1}(x) - \int f(y) dy \quad y = f^{-1}(x) \quad (4)$$

to evaluate the integrals in Exercises 77–80. Express your answers in terms of x .

$$77. \int \operatorname{arcsec} x dx$$

$$78. \int \arctan x dx$$

$$79. \int \sec^{-1} x dx$$

$$80. \int \log_2 x dx$$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x) dx = x f^{-1}(x) - \int x \left(\frac{d}{dx} f^{-1}(x) \right) dx. \quad (5)$$

Exercises 81 and 82 compare the results of using Equations (4) and (5).

81. Equations (4) and (5) give different formulas for the integral of $\arccos x$:

$$\text{a. } \int \arccos x dx = x \arccos x - \sin(\arccos x) + C \quad \text{Eq. (4)}$$

$$\text{b. } \int \arccos x dx = x \arccos x - \sqrt{1-x^2} + C \quad \text{Eq. (5)}$$

Can both integrations be correct? Explain.

82. Equations (4) and (5) lead to different formulas for the integral of $\arctan x$:

$$\text{a. } \int \arctan x dx = x \arctan x - \ln \sec(\arctan x) + C \quad \text{Eq. (4)}$$

$$\text{b. } \int \arctan x dx = x \arctan x - \ln \sqrt{1+x^2} + C \quad \text{Eq. (5)}$$

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 83 and 84 with (a) Eq. (4) and (b) Eq. (5). In each case, check your work by differentiating your answer with respect to x .

$$83. \int \sinh^{-1} x dx$$

$$84. \int \tanh^{-1} x dx$$

8.3 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x \, dx = \tan x + C.$$

The general idea is to use identities to transform the integrals we must find into integrals that are easier to work with.

Products of Powers of Sines and Cosines

We begin with integrals of the form

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If m is odd in $\int \sin^m x \cos^n x \, dx$, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \quad (1)$$

Then we substitute $u = \cos x$ and $du = -\sin x \, dx$.

Case 2 If n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then substitute $u = \sin x$ and $du = \cos x \, dx$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2} \quad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

EXAMPLE 1 Evaluate $\int \sin^3 x \cos^2 x \, dx$.

Solution This is an example of Case 1.

$$\begin{aligned} \int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx && m \text{ is odd.} \\ &= \int (1 - \cos^2 x)(\cos^2 x) \sin x \, dx && \sin^2 x = 1 - \cos^2 x \\ &= \int (1 - u^2)(u^2)(-du) && u = \cos x, \, du = -\sin x \, dx \\ &= \int (u^4 - u^2) \, du && \text{Distribute.} \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C \end{aligned}$$

EXAMPLE 2 Evaluate

$$\int \cos^5 x \, dx.$$

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\begin{aligned} \int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx \\ &= \int (1 - \sin^2 x)^2 \cos x \, dx \\ &= \int (1 - u^2)^2 \, du && u = \sin x, \, du = \cos x \, dx \\ &= \int (1 - 2u^2 + u^4) \, du && \text{Square } 1 - u^2. \\ &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C \end{aligned}$$

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution This is an example of Case 3.

$$\begin{aligned} \int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx && m \text{ and } n \text{ both even} \\ &= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) \, dx \\ &= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) \, dx \\ &= \frac{1}{8} \left[x + \frac{1}{2} \sin 2x - \int (\cos^2 2x + \cos^3 2x) \, dx \right] \end{aligned}$$

For the term involving $\cos^2 2x$, we use

$$\begin{aligned} \int \cos^2 2x \, dx &= \frac{1}{2} \int (1 + \cos 4x) \, dx && \text{Use the identity} \\ &= \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right) + C_1. && \cos^2 \theta = (1 + \cos 2\theta)/2, \\ & && \text{with } \theta = 2x. \end{aligned}$$

For the $\cos^3 2x$ term, we have

$$\begin{aligned} \int \cos^3 2x \, dx &= \int (1 - \sin^2 2x) \cos 2x \, dx && u = \sin 2x, \, du = 2 \cos 2x \, dx \\ &= \frac{1}{2} \int (1 - u^2) \, du = \frac{1}{2} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right) + C_2. \end{aligned}$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16} \left(x - \frac{1}{4} \sin 4x + \frac{1}{3} \sin^3 2x \right) + C.$$

Eliminating Square Roots

In the next example, we use the identity $\cos^2 \theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx.$$

Solution To eliminate the square root, we use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \text{or} \quad 1 + \cos 2\theta = 2 \cos^2 \theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2 \cos^2 2x.$$

Therefore,

$$\begin{aligned} \int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx &= \int_0^{\pi/4} \sqrt{2 \cos^2 2x} \, dx = \int_0^{\pi/4} \sqrt{2} \sqrt{\cos^2 2x} \, dx \\ &= \sqrt{2} \int_0^{\pi/4} |\cos 2x| \, dx = \sqrt{2} \int_0^{\pi/4} \cos 2x \, dx \quad \begin{array}{l} \cos 2x \geq 0 \\ \text{on } [0, \pi/4] \end{array} \\ &= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} = \frac{\sqrt{2}}{2} [1 - 0] = \frac{\sqrt{2}}{2}. \end{aligned}$$

Integrals of Powers of $\tan x$ and $\sec x$

We know how to integrate the tangent and secant functions and their squares. To integrate higher powers, we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE 5 Evaluate

$$\int \tan^4 x \, dx.$$

Solution

$$\begin{aligned} \int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx \\ &= \int \tan^2 x \cdot (\sec^2 x - 1) \, dx \quad \tan^2 x = \sec^2 x - 1 \\ &= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \quad \tan^2 x = \sec^2 x - 1 \\ &= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx \end{aligned}$$

In the first integral, we let

$$u = \tan x, \quad du = \sec^2 x \, dx$$

and have

$$\int u^2 du = \frac{1}{3}u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x dx = \frac{1}{3}\tan^3 x - \tan x + x + C. \quad \blacksquare$$

EXAMPLE 6 Evaluate

$$\int \sec^3 x dx.$$

Solution We integrate by parts using

$$u = \sec x, \quad dv = \sec^2 x dx, \quad v = \tan x, \quad du = \sec x \tan x dx.$$

Then

$$\begin{aligned} \int \sec^3 x dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x) dx && \text{Integrate by parts.} \\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x dx && \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \end{aligned}$$

Combining the two secant-cubed integrals gives

$$2 \int \sec^3 x dx = \sec x \tan x + \int \sec x dx$$

and therefore

$$\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C. \quad \blacksquare$$

EXAMPLE 7 Evaluate

$$\int \tan^4 x \sec^4 x dx.$$

Solution

$$\begin{aligned} \int (\tan^4 x)(\sec^4 x) dx &= \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x) dx && \sec^2 x = 1 + \tan^2 x \\ &= \int (\tan^4 x + \tan^6 x)(\sec^2 x) dx && \text{Distribute.} \\ &= \int (u^4 + u^6) du = \frac{u^5}{5} + \frac{u^7}{7} + C && u = \tan x, \\ &= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C && du = \sec^2 x dx \end{aligned} \quad \blacksquare$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \quad \int \sin mx \cos nx \, dx, \quad \text{and} \quad \int \cos mx \cos nx \, dx$$

arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the following identities.

$$\sin mx \sin nx = \frac{1}{2}[\cos(m - n)x - \cos(m + n)x] \quad (3)$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m - n)x + \sin(m + n)x] \quad (4)$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m - n)x + \cos(m + n)x] \quad (5)$$

These identities come from the angle sum formulas for the sine and cosine functions (Section 1.3). They give functions whose antiderivatives are easily found.

EXAMPLE 8 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned} \int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int [\sin(-2x) + \sin 8x] \, dx \\ &= \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C. \end{aligned}$$

EXERCISES 8.3

Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–22.

1. $\int \cos 2x \, dx$

2. $\int_0^{\pi} 3 \sin \frac{x}{3} \, dx$

11. $\int \sin^3 x \cos^3 x \, dx$

12. $\int \cos^3 2x \sin^5 2x \, dx$

3. $\int \cos^3 x \sin x \, dx$

4. $\int \sin^4 2x \cos 2x \, dx$

13. $\int \cos^2 x \, dx$

14. $\int_0^{\pi/2} \sin^2 x \, dx$

5. $\int \sin^3 x \, dx$

6. $\int \cos^3 4x \, dx$

15. $\int_0^{\pi/2} \sin^7 y \, dy$

16. $\int 7 \cos^7 t \, dt$

7. $\int \sin^5 x \, dx$

8. $\int_0^{\pi} \sin^5 \frac{x}{2} \, dx$

17. $\int_0^{\pi} 8 \sin^4 x \, dx$

18. $\int 8 \cos^4 2\pi x \, dx$

9. $\int \cos^3 x \, dx$

10. $\int_0^{\pi/6} 3 \cos^5 3x \, dx$

19. $\int 16 \sin^2 x \cos^2 x \, dx$

20. $\int_0^{\pi} 8 \sin^4 y \cos^2 y \, dy$

21. $\int 8 \cos^3 2\theta \sin 2\theta \, d\theta$

22. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$

Integrating Square Roots

Evaluate the integrals in Exercises 23–32.

23. $\int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}} dx$

24. $\int_0^\pi \sqrt{1 - \cos 2x} dx$

25. $\int_0^\pi \sqrt{1 - \sin^2 t} dt$

26. $\int_0^\pi \sqrt{1 - \cos^2 \theta} d\theta$

27. $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1 - \cos x}} dx$

28. $\int_0^{\pi/6} \sqrt{1 + \sin x} dx$

(Hint: Multiply by $\sqrt{\frac{1 - \sin x}{1 - \sin x}}$.)

29. $\int_{5\pi/6}^\pi \frac{\cos^4 x}{\sqrt{1 - \sin x}} dx$

30. $\int_{\pi/2}^{3\pi/4} \sqrt{1 - \sin 2x} dx$

31. $\int_0^{\pi/2} \theta \sqrt{1 - \cos 2\theta} d\theta$

32. $\int_{-\pi}^\pi (1 - \cos^2 t)^{3/2} dt$

Powers of Tangents and Secants

Evaluate the integrals in Exercises 33–52.

33. $\int \sec^2 x \tan x dx$

34. $\int \sec x \tan^2 x dx$

35. $\int \sec^3 x \tan x dx$

36. $\int \sec^3 x \tan^3 x dx$

37. $\int \sec^2 x \tan^2 x dx$

38. $\int \sec^4 x \tan^2 x dx$

39. $\int_{-\pi/3}^0 2 \sec^3 x dx$

40. $\int e^x \sec^3 e^x dx$

41. $\int \sec^4 \theta d\theta$

42. $\int \tan^4 x \sec^3 x dx$

43. $\int_{\pi/4}^{\pi/2} \csc^4 \theta d\theta$

44. $\int \sec^6 x dx$

45. $\int 4 \tan^3 x dx$

46. $\int_{-\pi/4}^{\pi/4} 6 \tan^4 x dx$

47. $\int \tan^5 x dx$

48. $\int \cot^6 2x dx$

49. $\int_{\pi/6}^{\pi/3} \cot^3 x dx$

50. $\int 8 \cot^4 t dt$

51. $\int_{\pi/4}^{\pi/3} \tan^5 \theta \sec^4 \theta d\theta$

52. $\int \cot^3 t \csc^4 t dt$

Products of Sines and Cosines

Evaluate the integrals in Exercises 53–58.

53. $\int \sin 3x \cos 2x dx$

54. $\int \sin 2x \cos 3x dx$

55. $\int_{-\pi}^\pi \sin 3x \sin 3x dx$

56. $\int_0^{\pi/2} \sin x \cos x dx$

57. $\int \cos 3x \cos 4x dx$

58. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x dx$

Exercises 59–64 require the use of various trigonometric identities before you evaluate the integrals.

59. $\int \sin^2 \theta \cos 3\theta d\theta$

60. $\int \cos^2 2\theta \sin \theta d\theta$

61. $\int \cos^3 \theta \sin 2\theta d\theta$

62. $\int \sin^3 \theta \cos 2\theta d\theta$

63. $\int \sin \theta \cos \theta \cos 3\theta d\theta$

64. $\int \sin \theta \sin 2\theta \sin 3\theta d\theta$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 65–70.

65. $\int \frac{\sec^3 x}{\tan x} dx$

66. $\int \frac{\sin^3 x}{\cos^4 x} dx$

67. $\int \frac{\tan^2 x}{\csc x} dx$

68. $\int \frac{\cot x}{\cos^2 x} dx$

69. $\int x \sin^2 x dx$

70. $\int x \cos^3 x dx$

Applications**71. Arc length** Find the length of the curve

$$y = \ln(\sin x), \quad \frac{\pi}{6} \leq x \leq \frac{\pi}{2}$$

72. Center of gravity Find the center of gravity of the region bounded by the x -axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.**73. Volume** Find the volume generated by revolving one arch of the curve $y = \sin x$ about the x -axis.**74. Area** Find the area between the x -axis and the curve $y = \sqrt{1 + \cos 4x}$, $0 \leq x \leq \pi$.**75. Centroid** Find the centroid of the region bounded by the graphs of $y = x + \cos x$ and $y = 0$ for $0 \leq x \leq 2\pi$.**76. Volume** Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sin x + \sec x$, $y = 0$, $x = 0$, and $x = \pi/3$ about the x -axis.**77. Volume** Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \arctan x$, $x = 0$, and $y = \pi/4$ about the y -axis.**78. Average Value** Find the average value of the function

$$f(x) = \frac{1}{1 - \sin \theta} \text{ on } [0, \pi/6].$$

64. What is the largest value that

$$\int_a^b x\sqrt{2x - x^2} dx$$

can have for any a and b ? Give reasons for your answer.

COMPUTER EXPLORATIONS

In Exercises 65 and 66, use a CAS to perform the integrations.

65. Evaluate the integrals

a. $\int x \ln x dx$ b. $\int x^2 \ln x dx$ c. $\int x^3 \ln x dx$.

d. What pattern do you see? Predict the formula for $\int x^4 \ln x dx$ and then see if you are correct by evaluating it with a CAS.

e. What is the formula for $\int x^n \ln x dx$, $n \geq 1$? Check your answer using a CAS.

66. Evaluate the integrals

a. $\int \frac{\ln x}{x^2} dx$ b. $\int \frac{\ln x}{x^3} dx$ c. $\int \frac{\ln x}{x^4} dx$.

d. What pattern do you see? Predict the formula for

$$\int \frac{\ln x}{x^5} dx$$

and then see if you are correct by evaluating it with a CAS.

e. What is the formula for

$$\int \frac{\ln x}{x^n} dx, \quad n \geq 2?$$

Check your answer using a CAS.

67. a. Use a CAS to evaluate

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx,$$

where n is an arbitrary positive integer. Does your CAS find the result?

b. In succession, find the integral when $n = 1, 2, 3, 5$, and 7 . Comment on the complexity of the results.

c. Now substitute $x = (\pi/2) - u$ and add the new and old integrals. What is the value of

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx?$$

This exercise illustrates how a little mathematical ingenuity can sometimes solve a problem not immediately amenable to solution by a CAS.

8.7 Numerical Integration

The antiderivatives of some functions, like $\sin(x^2)$, $1/\ln x$, and $\sqrt{1+x^4}$, have no elementary formulas. When we cannot find a workable antiderivative for a function f that we have to integrate, we can partition the interval of integration, replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the definite integral of f . This procedure is an example of numerical integration. In this section we start by revisiting the Midpoint Rule, which we studied in Section 5.2. We then study two new methods, the *Trapezoidal Rule* and *Simpson's Rule*. A key goal in our analysis is to control the possible error that is introduced when computing an approximation to an integral.

Approximating Integrals with the Midpoint Rule

In Section 5.2 we introduced the Midpoint Rule to approximate a definite integral over an interval $[a, b]$. The rule is based on subdividing $[a, b]$ into n equal subintervals,

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$$

each of width

$$\Delta x = \frac{b-a}{n}.$$

We then approximate the integral using n rectangles, where the height of the k th rectangle is the value of f at the midpoint $c_k = (x_{k-1} + x_k)/2$ of the k th subinterval $[x_{k-1}, x_k]$.

Midpoint Rule for Approximating a Definite Integral

$$\int_a^b f(x) dx \approx \sum_{k=1}^n f(c_k) \left(\frac{b-a}{n} \right) = [f(c_1) + f(c_2) + \dots + f(c_n)] \left(\frac{b-a}{n} \right)$$

$$\text{with } c_k = \frac{x_{k-1} + x_k}{2} \text{ and } x_k = a + k \left(\frac{b-a}{n} \right).$$

Trapezoidal Approximations

The Trapezoidal Rule for the value of a definite integral is based on approximating the region between a curve and the x -axis with trapezoids instead of rectangles, as in Figure 8.7. It is not necessary for the subdivision points $x_0, x_1, x_2, \dots, x_n$ in the figure to be evenly spaced, but the resulting formula is simpler if they are. We therefore assume that the length of each subinterval is

$$\Delta x = \frac{b - a}{n}.$$

The length $\Delta x = (b - a)/n$ is called the **step size** or **mesh size**. The area of the trapezoid that lies above the i th subinterval is

$$\Delta x \left(\frac{y_{i-1} + y_i}{2} \right) = \frac{\Delta x}{2} (y_{i-1} + y_i),$$

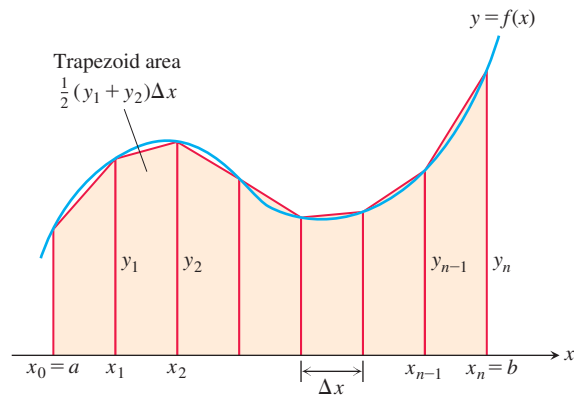


FIGURE 8.7 The Trapezoidal Rule approximates short stretches of the curve $y = f(x)$ with line segments. To approximate the integral of f from a to b , we add the areas of the trapezoids made by vertically joining the ends of the segments to the x -axis.

where $y_{i-1} = f(x_{i-1})$ and $y_i = f(x_i)$. (See Figure 8.7.) The area below the curve $y = f(x)$ and above the x -axis is then approximated by adding the areas of all the trapezoids:

$$\begin{aligned} T &= \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x + \cdots \\ &\quad + \frac{1}{2}(y_{n-2} + y_{n-1})\Delta x + \frac{1}{2}(y_{n-1} + y_n)\Delta x \\ &= \Delta x \left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n \right) \\ &= \frac{\Delta x}{2} (y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n), \end{aligned}$$

where

$$y_0 = f(a), \quad y_1 = f(x_1), \quad \dots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(b).$$

The Trapezoidal Rule says: Use T to estimate the integral of f from a to b .

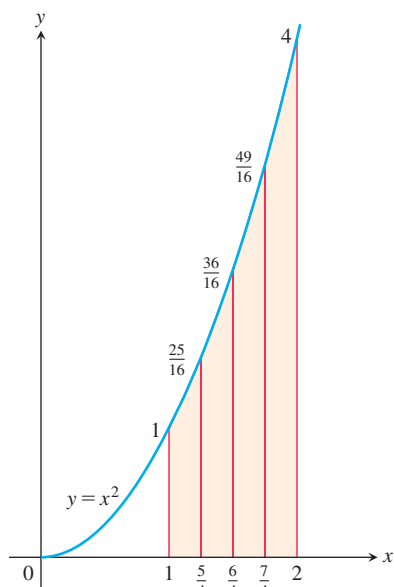


FIGURE 8.8 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate (Example 1).

TABLE 8.2

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

The Trapezoidal Rule

To approximate $\int_a^b f(x) dx$, use

$$T = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n).$$

The y 's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \dots, x_{n-1} = a + (n-1)\Delta x, x_n = b,$$

where $\Delta x = (b - a)/n$.

EXAMPLE 1 Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2 dx$. Compare the estimate with the exact value.

Solution Partition $[1, 2]$ into four subintervals of equal length (Figure 8.8). Then evaluate $y = x^2$ at each partition point (Table 8.2).

Using these y -values, $n = 4$, and $\Delta x = (2 - 1)/4 = 1/4$ in the Trapezoidal Rule, we have

$$\begin{aligned} T &= \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) \\ &= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) \\ &= \frac{75}{32} = 2.34375. \end{aligned}$$

Since the parabola is concave up, the approximating segments lie above the curve, giving each trapezoid slightly more area than the corresponding strip under the curve. The exact value of the integral is

$$\int_1^2 x^2 dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

The T approximation overestimates the integral by about half a percent of its true value of $7/3$. The percentage error is $(2.34375 - 7/3)/(7/3) \approx 0.00446$, or 0.446%. ■

Simpson's Rule: Approximations Using Parabolas

Another rule for approximating the definite integral of a continuous function results from using parabolas instead of the straight-line segments that produced trapezoids. As before, we partition the interval $[a, b]$ into n subintervals of equal length $h = \Delta x = (b - a)/n$, but this time we require that n be an *even* number. On each consecutive pair of intervals we approximate the curve $y = f(x) \geq 0$ by a parabola, as shown in Figure 8.9. A typical parabola passes through three consecutive points (x_{i-1}, y_{i-1}) , (x_i, y_i) , and (x_{i+1}, y_{i+1}) on the curve.

Let's calculate the shaded area beneath a parabola passing through three consecutive points. To simplify our calculations, we first take the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$ (Figure 8.10), where $h = \Delta x = (b - a)/n$. The area under the parabola will be the same if we shift the y -axis to the left or right. The parabola has an equation of the form

$$y = Ax^2 + Bx + C,$$

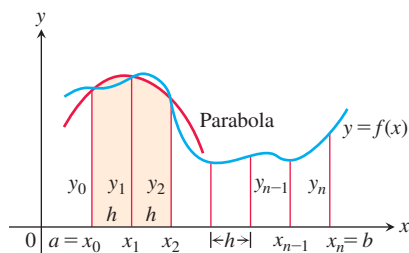


FIGURE 8.9 Simpson's Rule approximates short stretches of the curve with parabolas.

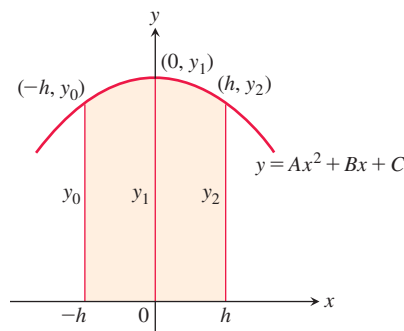


FIGURE 8.10 By integrating from $-h$ to h , we find the shaded area to be

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

so the area under it from $x = -h$ to $x = h$ is

$$\begin{aligned} A_p &= \int_{-h}^h (Ax^2 + Bx + C) dx \\ &= \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \right]_{-h}^h \\ &= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C). \end{aligned}$$

Since the curve passes through the three points $(-h, y_0)$, $(0, y_1)$, and (h, y_2) , we also have

$$y_0 = Ah^2 - Bh + C, \quad y_1 = C, \quad y_2 = Ah^2 + Bh + C.$$

After some algebraic manipulation, we find that

$$A_p = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Now shifting the parabola horizontally to its shaded position in Figure 8.9 does not change the area under it. Thus the area under the parabola through (x_0, y_0) , (x_1, y_1) , and (x_2, y_2) in Figure 8.9 is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

Similarly, the area under the parabola through the points (x_2, y_2) , (x_3, y_3) , and (x_4, y_4) is

$$\frac{h}{3}(y_2 + 4y_3 + y_4).$$

Computing the areas under all the parabolas and adding the results give the approximation

$$\begin{aligned} \int_a^b f(x) dx &\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots \\ &\quad + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n). \end{aligned}$$

HISTORICAL BIOGRAPHY

Thomas Simpson
(1720–1761)

Simpson was a successful text writer and did most of his research on probability. Simpson's rule to approximate definite integrals was developed before he was born. It is another of history's beautiful quirks that one of the ablest mathematicians of the 18th century is remembered not for his own work but for a rule that was never his, that he never claimed, and that bears his name only because he happened to mention it in one of his books.

To know more, visit the companion Website.

The result is known as Simpson's Rule. The function need not be positive, as in our derivation, but the number n of subintervals must be even for us to apply the rule because each parabolic arc uses two subintervals.

Simpson's Rule

To approximate $\int_a^b f(x) dx$, use

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

The y 's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \dots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

The number n is even, and $\Delta x = (b - a)/n$.

Note the pattern of the coefficients in the above rule: 1, 4, 2, 4, 2, 4, 2, ..., 4, 1.

EXAMPLE 2 Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4 dx$.

TABLE 8.3

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

Solution Partition $[0, 2]$ into four subintervals and evaluate $y = 5x^4$ at the partition points (Table 8.3). Then apply Simpson's Rule with $n = 4$ and $\Delta x = 1/2$:

$$\begin{aligned}
 S &= \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \\
 &= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right) \\
 &= 32\frac{1}{12}.
 \end{aligned}$$

This estimate differs from the exact value (32) by only $1/12$, a percentage error of less than three-tenths of one percent, and this was with just four subintervals. ■

Error Analysis

Whenever we use an approximation technique, we must consider how accurate the approximation might be. The following theorem gives formulas for estimating the errors when using the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. The **error** is the difference between the approximation obtained by using the rule and the actual value of the definite integral $\int_a^b f(x) dx$.

THEOREM 1—Error Estimates in the Midpoint, Trapezoidal, and Simpson's Rules

If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_M in the Midpoint Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_M| \leq \frac{M(b-a)^3}{24n^2}. \quad \text{Midpoint Rule}$$

If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the Trapezoidal Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}. \quad \text{Trapezoidal Rule}$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}. \quad \text{Simpson's Rule}$$

To give an idea of why Theorem 1 is true in the case of the Trapezoidal Rule, we begin with a result which says that if f'' is continuous on the interval $[a, b]$, then

$$\int_a^b f(x) dx = T - \frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

for some number c between a and b . This result follows from Taylor's Remainder Theorem, which we discuss in Theorem 24 of Section 9.9. It follows that as Δx approaches zero, the error defined by

$$E_T = -\frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

approaches zero at the rate of the square of Δx .

The inequality

$$|E_T| \leq \frac{b-a}{12} \max |f''(x)| (\Delta x)^2,$$

where “max” refers to the maximum of $|f''(x)|$ over the interval $[a, b]$, gives an upper bound for the magnitude of the error. In practice, we usually cannot find the exact value of $\max |f''(x)|$ and have to estimate an upper bound or “worst case” value for it instead. If M is any upper bound for the values of $|f''(x)|$ on $[a, b]$, so that $|f''(x)| \leq M$ for every x in $[a, b]$, then

$$|E_T| \leq \frac{b-a}{12} M (\Delta x)^2.$$

If we substitute $(b-a)/n$ for Δx , we get

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

The upper bound for the error in the Midpoint Rule,

$$|E_M| \leq \frac{M(b-a)^3}{24n^2},$$

is based on a similar argument. Taylor’s Remainder Theorem implies that the Midpoint Rule approximation and the definite integral differ by at most

$$\frac{|f''(c)| (b-a)^3}{24n^2}$$

for some $c \in [a, b]$. Taking M to be at least as large as the maximum of $|f''(c)|$ on $[a, b]$, we obtain the error bound in Theorem 1. Note that this is half as large as the error bound for the Trapezoidal Rule. This does not mean that the Midpoint Rule is always more accurate than the Trapezoidal Rule, but rather that the largest possible error that might occur is only half as large as the largest possible error for the Trapezoidal Rule.

To estimate the error in Simpson’s Rule, we start with a result, again following from Taylor’s Remainder Theorem, that says that if the fourth derivative $f^{(4)}$ is continuous, then

$$\int_a^b f(x) dx = S - \frac{b-a}{180} \cdot f^{(4)}(c) (\Delta x)^4$$

for some point c between a and b . Thus, as Δx approaches zero, the error,

$$E_S = -\frac{b-a}{180} \cdot f^{(4)}(c) (\Delta x)^4,$$

approaches zero as the *fourth power* of Δx . (This helps to explain why Simpson’s Rule is likely to give better results than the Trapezoidal Rule.)

The inequality

$$|E_S| \leq \frac{b-a}{180} \max |f^{(4)}(x)| (\Delta x)^4,$$

where “max” refers to the maximum of $|f^{(4)}(x)|$ over the interval $[a, b]$, gives an upper bound for the magnitude of the error. As with $\max |f''|$ in the error formula for the Trapezoidal Rule, we usually cannot find the exact value of $\max |f^{(4)}(x)|$ and have to replace it with an upper bound. If M is any upper bound for the values of $|f^{(4)}(x)|$ on $[a, b]$, then

$$|E_S| \leq \frac{b-a}{180} M (\Delta x)^4.$$

Substituting $(b-a)/n$ for Δx in this last expression gives

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

You might wonder why we don't just take M to be the maximum value of $|f''(x)|$ on $[a, b]$ for the first two rules, or the maximum value of $|f^{(4)}(x)|$ for Simpson's Rule. The reason is that sometimes this maximum is hard to compute, while a less accurate upper bound is easily found. We can certainly set M to the maximum value if we can compute it.

EXAMPLE 3 Find an upper bound for the error in estimating $\int_0^2 5x^4 dx$ using Simpson's Rule with $n = 4$ (Example 2).

Solution To estimate the error, we first find an upper bound M for the magnitude of the fourth derivative of $f(x) = 5x^4$ on the interval $0 \leq x \leq 2$. Since the fourth derivative has the constant value $f^{(4)}(x) = 120$, we take $M = 120$. With $b - a = 2$ and $n = 4$, the error estimate for Simpson's Rule gives

$$|E_s| \leq \frac{M(b-a)^5}{180n^4} = \frac{120(2)^5}{180 \cdot 4^4} = \frac{1}{12}.$$

This estimate is consistent with the result of Example 2. ■

Theorem 1 can also be used to estimate the number of subintervals required when using the Trapezoidal or Simpson's Rule if we specify a certain tolerance for the error.

EXAMPLE 4 Estimate the minimum number of subintervals needed to approximate the integral in Example 3 using Simpson's Rule with an error of magnitude less than 10^{-4} .

Solution Using the inequality in Theorem 1, if we choose the number of subintervals n to satisfy

$$\frac{M(b-a)^5}{180n^4} < 10^{-4},$$

then the error E_s in Simpson's Rule satisfies $|E_s| < 10^{-4}$, as required.

From the solution in Example 3, we have $M = 120$ and $b - a = 2$, so we want n to satisfy

$$\frac{120(2)^5}{180n^4} < \frac{1}{10^4},$$

or, equivalently,

$$n^4 > \frac{64 \cdot 10^4}{3}.$$

It follows that

$$n > 10 \left(\frac{64}{3} \right)^{1/4} \approx 21.5.$$

Since n must be even in Simpson's Rule, we estimate the minimum number of subintervals required for the error tolerance to be $n = 22$. ■

EXAMPLE 5 As we saw in Chapter 7, the value of $\ln 2$ can be calculated from the integral

$$\ln 2 = \int_1^2 \frac{1}{x} dx.$$

Table 8.4 shows values of T and S for approximations of $\int_1^2 (1/x) dx$ using various values of n . Notice how Simpson's Rule dramatically improves over the Trapezoidal Rule.

TABLE 8.4 Trapezoidal Rule approximations (T_n) and Simpson's Rule approximations (S_n) of $\ln 2 = \int_1^2 (1/x) dx$

n	T_n	Error less than ...	S_n	Error less than ...
10	0.6937714032	0.0006242227	0.6931502307	0.0000030502
20	0.6933033818	0.0001562013	0.6931473747	0.0000001942
30	0.6932166154	0.0000694349	0.6931472190	0.0000000385
40	0.6931862400	0.0000390595	0.6931471927	0.0000000122
50	0.6931721793	0.0000249988	0.6931471856	0.0000000050
100	0.6931534305	0.0000062500	0.6931471809	0.0000000004

In particular, notice that when we double the value of n (thereby halving the value of $h = \Delta x$), the T error is divided by 2 *squared*, whereas the S error is divided by 2 *to the fourth*.

This has a dramatic effect as $\Delta x = (2 - 1)/n$ gets very small. The Simpson approximation for $n = 50$ rounds accurately to seven places and for $n = 100$ is accurate to nine decimal places (billionths)! ■

If $f(x)$ is a polynomial of degree less than 4, then its fourth derivative is zero, and

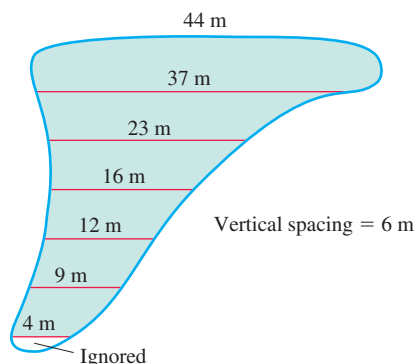
$$E_S = -\frac{b-a}{180} f^{(4)}(c)(\Delta x)^4 = -\frac{b-a}{180} (0)(\Delta x)^4 = 0.$$

Thus, there will be no error in the Simpson approximation of any integral of f . In other words, if f is a constant, a linear function, or a quadratic or cubic polynomial, Simpson's Rule will give the value of any integral of f exactly, whatever the number of subdivisions. Similarly, if f is a constant or a linear function, then its second derivative is zero, and

$$E_T = -\frac{b-a}{12} f''(c)(\Delta x)^2 = -\frac{b-a}{12} (0)(\Delta x)^2 = 0.$$

The Trapezoidal Rule will therefore give the exact value of any integral of f . This is no surprise, for the trapezoids fit the graph perfectly.

Although decreasing the step size Δx reduces the error in the Simpson and Trapezoidal approximations in theory, it may fail to do so in practice. When Δx is very small, say $\Delta x = 10^{-8}$, computer or calculator round-off errors in the arithmetic required to evaluate S and T may accumulate to such an extent that the error formulas no longer describe what is going on. Shrinking Δx below a certain size can actually make things worse. You should consult a text on numerical analysis for more sophisticated methods if you are having problems with round-off error using the rules discussed in this section.

**FIGURE 8.11** The dimensions of the swamp in Example 6.

EXAMPLE 6 A town wants to drain and fill a polluted swamp (Figure 8.11). The swamp averages 1.5 m deep. About how many cubic meters of dirt will it take to fill the area after the swamp is drained?

Solution To calculate the volume of the swamp, we estimate the surface area and multiply by 1.5. To estimate the area, we use Simpson's Rule with $\Delta x = 6$ m, and the y 's equal to the distances measured across the swamp, as shown in Figure 8.11.

$$\begin{aligned} S &= \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6) \\ &= \frac{6}{3} (44 + 148 + 46 + 64 + 24 + 36 + 4) = 732 \end{aligned}$$

The volume is about $(732)(1.5) = 1098 \text{ m}^3$. ■

EXERCISES
8.7

For some exercises, a calculator may be helpful for expressing answers in decimal form.

Estimating Definite Integrals

The instructions for the integrals in Exercises 1–10 have three parts, one for the Midpoint Rule, one for the Trapezoidal Rule, and one for Simpson’s Rule.

I. Using the Midpoint Rule

- a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_M|$.
- b. Evaluate the integral directly and find $|E_M|$.
- c. Use the formula $(|E_M|/(\text{true value})) \times 100$ to express $|E_M|$ as a percentage of the integral’s true value.

II. Using the Trapezoidal Rule

- a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_T|$.
- b. Evaluate the integral directly and find $|E_T|$.
- c. Use the formula $(|E_T|/(\text{true value})) \times 100$ to express $|E_T|$ as a percentage of the integral’s true value.

III. Using Simpson’s Rule

- a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_S|$.
- b. Evaluate the integral directly and find $|E_S|$.
- c. Use the formula $(|E_S|/(\text{true value})) \times 100$ to express $|E_S|$ as a percentage of the integral’s true value.

1. $\int_1^2 x \, dx$
2. $\int_1^3 (2x - 1) \, dx$
3. $\int_{-1}^1 (x^2 + 1) \, dx$
4. $\int_{-2}^0 (x^2 - 1) \, dx$
5. $\int_0^2 (t^3 + t) \, dt$
6. $\int_{-1}^1 (t^3 + 1) \, dt$
7. $\int_1^2 \frac{1}{s^2} \, ds$
8. $\int_2^4 \frac{1}{(s - 1)^2} \, ds$
9. $\int_0^\pi \sin t \, dt$
10. $\int_0^1 \sin \pi t \, dt$

Estimating the Number of Subintervals

In Exercises 11–22, estimate the minimum number of subintervals needed to approximate the integrals with an error of magnitude less than 10^{-4} by (a) the Trapezoidal Rule and (b) Simpson’s Rule. (The integrals in Exercises 11–18 are the integrals from Exercises 1–8.)

11. $\int_1^2 x \, dx$
12. $\int_1^3 (2x - 1) \, dx$
13. $\int_{-1}^1 (x^2 + 1) \, dx$
14. $\int_{-2}^0 (x^2 - 1) \, dx$
15. $\int_0^2 (t^3 + t) \, dt$
16. $\int_{-1}^1 (t^3 + 1) \, dt$

17. $\int_1^2 \frac{1}{s^2} \, ds$
18. $\int_2^4 \frac{1}{(s - 1)^2} \, ds$
19. $\int_0^3 \sqrt{x + 1} \, dx$
20. $\int_0^3 \frac{1}{\sqrt{x + 1}} \, dx$
21. $\int_0^2 \sin(x + 1) \, dx$
22. $\int_{-1}^1 \cos(x + \pi) \, dx$

Estimates with Numerical Data

23. **Volume of water in a swimming pool** A rectangular swimming pool is 5 m wide and 10 m long. The accompanying table shows the depth $h(x)$ of the water at 1-m intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$ applied to the integral

$$V = \int_0^{10} 5 \cdot h(x) \, dx.$$

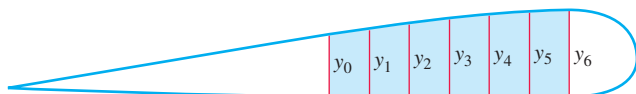
Position (m) x	Depth (m) $h(x)$	Position (m) x	Depth (m) $h(x)$
0	1.20	6	2.30
1	1.64	7	2.38
2	1.82	8	2.46
3	1.98	9	2.54
4	2.10	10	2.60
5	2.20		

24. **Distance traveled** The accompanying table shows time-to-speed data for a car accelerating from rest to 130 km/h. How far had the car traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: The time intervals vary in length.)

Speed change	Time (s)
Zero to 30 km/h	2.2
40 km/h	3.2
50 km/h	4.5
60 km/h	5.9
70 km/h	7.8
80 km/h	10.2
90 km/h	12.7
100 km/h	16.0
110 km/h	20.6
120 km/h	26.2
130 km/h	37.1

25. **Wing design** The design of a new airplane requires a gasoline tank of constant cross-sectional area in each wing. A scale drawing of a cross-section is shown here. The tank must hold 2000 kg

of gasoline, which has a density of 673 kg/m^3 . Estimate the length of the tank by Simpson's Rule.



$$y_0 = 0.5 \text{ m}, \quad y_1 = 0.55 \text{ m}, \quad y_2 = 0.6 \text{ m}, \quad y_3 = 0.65 \text{ m}, \\ y_4 = 0.7 \text{ m}, \quad y_5 = y_6 = 0.75 \text{ m} \quad \text{Horizontal spacing} = 0.3 \text{ m}$$

- 26. Oil consumption on Pathfinder Island** A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced. Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

Day	Oil consumption rate (liters/hour)
Sun	0.019
Mon	0.020
Tue	0.021
Wed	0.023
Thu	0.025
Fri	0.028
Sat	0.031
Sun	0.035

Theory and Examples

- 27. Usable values of the sine-integral function** The sine-integral function,

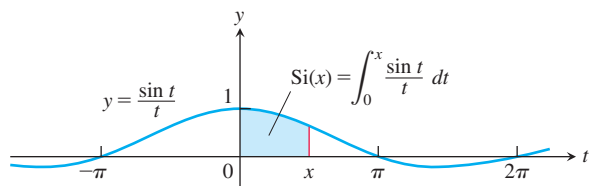
$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt, \quad \text{"Sine integral of } x\text{"}$$

is one of the many functions in engineering whose formulas cannot be simplified. There is no elementary formula for the antiderivative of $(\sin t)/t$. The values of $\text{Si}(x)$, however, are readily estimated by numerical integration.

Although the notation does not show it explicitly, the function being integrated is

$$f(t) = \begin{cases} \frac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the interval $[0, x]$. The function has derivatives of all orders at every point of its domain. Its graph is smooth, and you can expect good results from Simpson's Rule.



- a. Use the fact that $|f^{(4)}| \leq 1$ on $[0, \pi/2]$ to give an upper bound for the error that will occur if

$$\text{Si}\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin t}{t} dt$$

is estimated by Simpson's Rule with $n = 4$.

- b. Estimate $\text{Si}(\pi/2)$ by Simpson's Rule with $n = 4$.
c. Express the error bound you found in part (a) as a percentage of the value you found in part (b).
- 28. The error function** The error function,

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

which is important in probability and in the theories of heat flow and signal transmission, must be evaluated numerically because there is no elementary expression for the antiderivative of e^{-t^2} .

- a. Use Simpson's Rule with $n = 10$ to estimate $\text{erf}(1)$.
b. In $[0, 1]$,

$$\left| \frac{d^4}{dt^4}(e^{-t^2}) \right| \leq 12.$$

Give an upper bound for the magnitude of the error of the estimate in part (a).

- 29.** Prove that the sum T in the Trapezoidal Rule for $\int_a^b f(x) dx$ is a Riemann sum for f continuous on $[a, b]$. (Hint: Use the Intermediate Value Theorem to show the existence of c_k in the subinterval $[x_{k-1}, x_k]$ satisfying $f(c_k) = (f(x_{k-1}) + f(x_k))/2$.)
30. Prove that the sum S in Simpson's Rule for $\int_a^b f(x) dx$ is a Riemann sum for f continuous on $[a, b]$. (See Exercise 29.)
31. Elliptic integrals The length of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

turns out to be

$$\text{Length} = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \cos^2 t} dt,$$

where $e = \sqrt{a^2 - b^2}/a$ is the ellipse's eccentricity. The integral in this formula, called an *elliptic integral*, is nonelementary except when $e = 0$ or 1 .

- a. Use the Trapezoidal Rule with $n = 10$ to estimate the length of the ellipse when $a = 1$ and $e = 1/2$.
b. Use the fact that the absolute value of the second derivative of $f(t) = \sqrt{1 - e^2 \cos^2 t}$ is less than 1 to find an upper bound for the error in the estimate you obtained in part (a).

Applications

- 32.** The length of one arch of the curve $y = \sin x$ is given by

$$L = \int_0^\pi \sqrt{1 + \cos^2 x} dx.$$

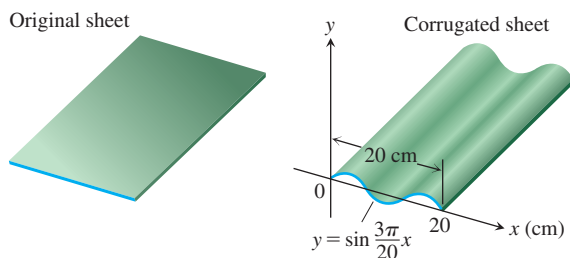
Estimate L by Simpson's Rule with $n = 8$.

T When solving Exercises 33–40, you may need to use a calculator or a computer.

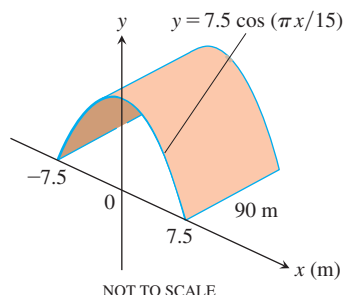
33. Your metal fabrication company is bidding for a contract to make sheets of corrugated iron roofing like the one shown here. The cross-sections of the corrugated sheets are to conform to the curve

$$y = \sin \frac{3\pi}{20}x, \quad 0 \leq x \leq 20 \text{ cm.}$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? To find out, use numerical integration to approximate the length of the sine curve to two decimal places.



34. Your engineering firm is bidding for the contract to construct the tunnel shown here. The tunnel is 90 m long and 15 m wide at the base. The cross-section is shaped like one arch of the curve $y = 7.5 \cos(\pi x/15)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$26.11 per square meter to apply. How much will it cost to apply the sealer? (*Hint*: Use numerical integration to find the length of the cosine curve.)



Find, to two decimal places, the areas of the surfaces generated by revolving the curves in Exercises 35 and 36 about the x -axis.

35. $y = \sin x, \quad 0 \leq x \leq \pi$

36. $y = x^2/4, \quad 0 \leq x \leq 2$

37. Use numerical integration to estimate the value of

$$\arcsin 0.6 = \int_0^{0.6} \frac{dx}{\sqrt{1-x^2}}.$$

For reference, $\arcsin 0.6 = 0.64350$ to five decimal places.

38. Use numerical integration to estimate the value of

$$\pi = 4 \int_0^1 \frac{1}{1+x^2} dx.$$

39. **Drug assimilation** An average adult under age 60 years assimilates a 12-hour cold medicine into his or her system at a rate modeled by

$$\frac{dy}{dt} = 6 - \ln(2t^2 - 3t + 3),$$

where y is measured in milligrams and t is the time in hours since the medication was taken. What amount of medicine is absorbed into a person's system over a 12-hour period?

40. **Effects of an antihistamine** The concentration of an antihistamine in the bloodstream of a healthy adult is modeled by

$$C = 12.5 - 4 \ln(t^2 - 3t + 4),$$

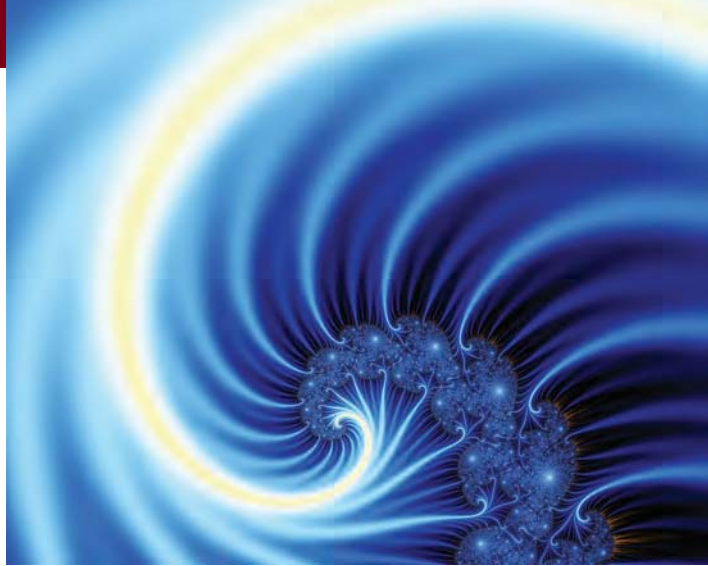
where C is measured in grams per liter and t is the time in hours since the medication was taken. What is the average level of concentration in the bloodstream over a 6-hour period?

8.8 Improper Integrals

Up to now, we have required definite integrals to satisfy two properties. First, the domain of integration $[a, b]$ must be finite. Second, the range of the integrand must be finite on this domain. In practice, we may encounter problems that fail to meet one or both of these conditions. The integral for the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ is an example for which the domain is infinite (Figure 8.12a). The integral for the area under the curve of $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ is an example for which the range of the integrand is infinite (Figure 8.12b). In either case, the integrals are said to be *improper* and are calculated as limits. We will see in Chapter 9 that improper integrals are useful for investigating the convergence of certain infinite series.

9

Infinite Sequences and Series



OVERVIEW In this chapter we introduce the topic of *infinite series*. Such series give us precise ways to express many numbers and functions, both familiar and new, as arithmetic sums with infinitely many terms. For example, we will learn that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

and

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40,320} - \cdots.$$

We need to develop a method to make sense of such expressions. Everyone knows how to add two numbers together, or even several. But how do you add together infinitely many numbers? Or, when adding together functions, how do you add infinitely many powers of x ? In this chapter we answer these questions, which are part of the theory of infinite sequences and series. As with the differential and integral calculus, limits play a major role in the development of infinite series.

One common and important application of series occurs in making computations with complicated functions. A hard-to-compute function is replaced by an expression that looks like an “infinite degree polynomial,” an infinite series in powers of x , as we see with the cosine function given above. Using the first few terms of this infinite series can allow for highly accurate approximations of functions by polynomials, enabling us to work with more general functions than those we have encountered before. These new functions are commonly obtained as solutions to differential equations arising in important applications of mathematics to science and engineering.

The terms “sequence” and “series” are sometimes used interchangeably in spoken language. In mathematics, however, each has a distinct meaning. A sequence is a type of infinite list, whereas a series is an infinite sum. To understand the infinite sums described by series, we first must understand infinite sequences.

9.1 Sequences

HISTORICAL ESSAY

Sequences and Series

To read this essay, visit the companion Website.

Sequences are fundamental to the study of infinite series and to many aspects of mathematics. We saw one example of a sequence when we studied Newton’s Method in Section 4.7. Newton’s Method produces a sequence of approximations x_n that become closer and closer to the root of a differentiable function. Now we will explore general sequences of numbers and the conditions under which they converge to a finite number.

Representing Sequences

A sequence is a list of numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

in a given order. Each of a_1, a_2, a_3 , and so on represents a number. These are the **terms** of the sequence. For example, the sequence

$$2, 4, 6, 8, 10, 12, \dots, 2n, \dots$$

has first term $a_1 = 2$, second term $a_2 = 4$, and n th term $a_n = 2n$. The integer n is called the **index** of a_n and indicates where a_n occurs in the list. Order is important. The sequence $2, 4, 6, 8, \dots$ is not the same as the sequence $4, 2, 6, 8, \dots$

We can think of the sequence

$$a_1, a_2, a_3, \dots, a_n, \dots$$

as a function that sends 1 to a_1 , 2 to a_2 , 3 to a_3 , and in general sends the positive integer n to the n th term a_n . More precisely, an **infinite sequence** of numbers is a function whose domain is the set of positive integers. For example, the function associated with the sequence

$$2, 4, 6, 8, 10, 12, \dots, 2n, \dots$$

sends 1 to $a_1 = 2$, 2 to $a_2 = 4$, and so on. The general behavior of this sequence is described by the formula $a_n = 2n$.

We can change the index to start at any given number n . For example, the sequence

$$12, 14, 16, 18, 20, 22, \dots$$

is described by the formula $a_n = 10 + 2n$, if we start with $n = 1$. It can also be described by the simpler formula $b_n = 2n$, where the index n starts at 6 and increases. To allow such simpler formulas, we let the first index of the sequence be any appropriate integer. In the sequence above, $\{a_n\}$ starts with a_1 while $\{b_n\}$ starts with b_6 .

Sequences can be described by writing rules that specify their terms, such as

$$a_n = \sqrt{n}, \quad b_n = (-1)^{n+1} \frac{1}{n}, \quad c_n = \frac{n-1}{n}, \quad d_n = (-1)^{n+1},$$

or by listing terms:

$$\begin{aligned} \{a_n\} &= \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\} \\ \{b_n\} &= \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1} \frac{1}{n}, \dots\right\} \\ \{c_n\} &= \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n-1}{n}, \dots\right\} \\ \{d_n\} &= \{1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}, \dots\}. \end{aligned}$$

We also sometimes write a sequence using its rule, as with

$$\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$$

and

$$\{b_n\} = \left\{(-1)^{n+1} \frac{1}{n}\right\}_{n=1}^{\infty}.$$

Figure 9.1 shows two ways to represent sequences graphically. The first marks the first few points from $a_1, a_2, a_3, \dots, a_n, \dots$ on the real axis. The second method shows the graph of the function defining the sequence. The function is defined only on integer inputs, and the graph consists of some points in the xy -plane located at $(1, a_1), (2, a_2), \dots, (n, a_n), \dots$

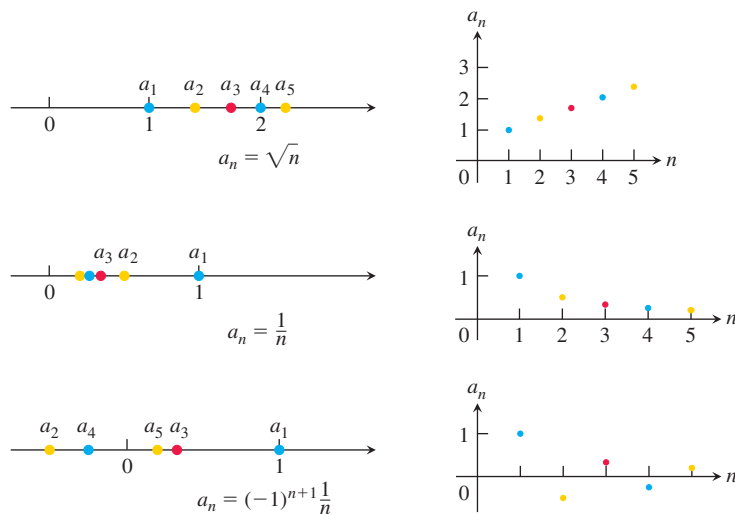


FIGURE 9.1 Sequences can be represented as points on the real line or as points in the plane where the horizontal axis n is the index number of the term and the vertical axis a_n is its value.

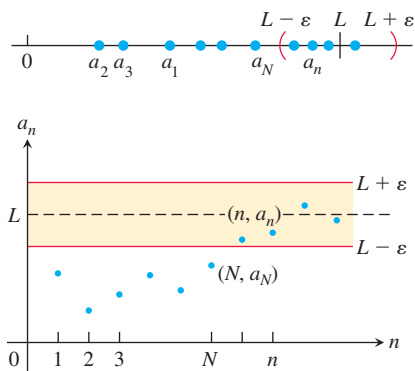


FIGURE 9.2 In the representation of a sequence as points in the plane, $a_n \rightarrow L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$. In this figure, all the a_n 's after a_N lie within ε of L .

Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index n increases. This happens in the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right\},$$

whose terms approach 0 as n gets large, and in the sequence

$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, 1 - \frac{1}{n}, \dots\right\},$$

whose terms approach 1. On the other hand, sequences like

$$\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\}$$

have terms that get larger than any number as n increases, and sequences like

$$\{1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}, \dots\}$$

bounce back and forth between 1 and -1 , never converging to a single value. The following definition captures the meaning of having a sequence converge to a limiting value. It says that if we specify any number $\varepsilon > 0$, then by going far enough out in the sequence, taking the index n to be larger than some value N , the difference between a_n and the limit of the sequence becomes less than ε .

HISTORICAL BIOGRAPHY

Nicole Oresme
(ca. 1320–1382)

Frenchman Oresme went to the University of Paris in the 1340s, studying theology and liberal arts. Later he was a faculty member and administrator at the same university. His work entitled *De configurationibus* (1350s) contained results in geometry and was the first to present graphs of velocities. The argument we use to show the divergence of the harmonic series was devised by Oresme in this publication.

To know more, visit the companion Website.

DEFINITIONS The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n > N.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L , we write $\lim_{n \rightarrow \infty} a_n = L$, or simply $a_n \rightarrow L$, and call L the **limit** of the sequence (Figure 9.2).

The definition is very similar to the definition of $\lim_{x \rightarrow \infty} f(x)$, the limit of a function $f(x)$ as x tends to ∞ , discussed in Section 2.5. We will exploit this connection to calculate limits of sequences.

EXAMPLE 1 Show that

$$(a) \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad (b) \lim_{n \rightarrow \infty} k = k \quad (\text{where } k \text{ is a constant})$$

Solution

(a) Let $\varepsilon > 0$ be given. We must show that there exists an integer N such that

$$\left| \frac{1}{n} - 0 \right| < \varepsilon \quad \text{whenever} \quad n > N.$$

The inequality $|1/n - 0| < \varepsilon$ will hold if $1/n < \varepsilon$ or $n > 1/\varepsilon$. If N is any integer greater than $1/\varepsilon$, the inequality will hold for all $n > N$. This proves that $\lim_{n \rightarrow \infty} 1/n = 0$.

(b) Let k be a constant, and let $\varepsilon > 0$ be given. We must show that there exists an integer N such that

$$|k - k| < \varepsilon \quad \text{whenever} \quad n > N.$$

Since $k - k = 0$, we can use any positive integer for N and the inequality $|k - k| < \varepsilon$ will hold. This proves that $\lim_{n \rightarrow \infty} k = k$. ■

EXAMPLE 2 Show that the sequence $\{1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}, \dots\}$ diverges.

Solution Since the terms in this sequence alternate between 1 and -1 , our intuition is that they cannot approach arbitrarily close to some single limit L . To prove this, suppose that there did exist a number L that is the limit of this sequence. The definition of convergence then tells us that for every number $\varepsilon > 0$, there is an integer N such that $|a_n - L| < \varepsilon$ for all $n > N$. No matter what positive number ε we consider, such an integer N must exist. In particular, for the number $\varepsilon = 1/2$, there must exist some integer N such that $|a_n - L| < 1/2$ for all $n > N$. If $n > N$ is odd, then $a_n = 1$, so we have $|1 - L| < 1/2$, or

$$\frac{1}{2} < L < \frac{3}{2}.$$

On the other hand, if $n > N$ is even, then $a_n = -1$, and so we have $|-1 - L| < 1/2$, or

$$-\frac{3}{2} < L < -\frac{1}{2}.$$

However, these two conditions on L cannot both hold, so we have reached a contradiction. Therefore there is no number L to which this sequence converges.

We chose $\varepsilon = 1/2$ in this argument simply for convenience. The same argument would work if we had instead chosen $\varepsilon = 1/4$, or any value less than 1. ■

The sequence $\{\sqrt{n}\}$ also diverges, but for a different reason. As n increases, its terms become larger than any fixed number. We describe the behavior of this sequence by writing

$$\lim_{n \rightarrow \infty} \sqrt{n} = \infty.$$

In writing infinity as the limit of a sequence, we are not saying that the differences between the terms a_n and ∞ become small as n increases, nor are we asserting that there is some number infinity that the sequence approaches. We are merely using a notation that captures the idea that a_n eventually gets and stays larger than any fixed number as n gets large (see Figure 9.3a). The terms of a sequence might also decrease to negative infinity, as in Figure 9.3b.

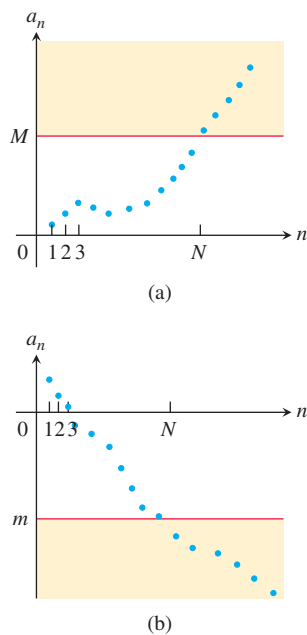


FIGURE 9.3 (a) The sequence diverges to ∞ because no matter what number M is chosen, the terms of the sequence after some index N all lie in the yellow band above M . (b) The sequence diverges to $-\infty$ because all terms after some index N lie below any chosen number m .

DEFINITION The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N , $a_n > M$. If this condition holds, then we write

$$\lim_{n \rightarrow \infty} a_n = \infty \quad \text{or} \quad a_n \rightarrow \infty.$$

Similarly, if for every number m there is an integer N such that for all $n > N$, we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n \rightarrow \infty} a_n = -\infty \quad \text{or} \quad a_n \rightarrow -\infty.$$

A sequence may diverge without diverging to infinity or negative infinity, as we saw in Example 2. The sequences $\{1, -2, 3, -4, 5, -6, 7, -8, \dots\}$ and $\{1, 0, 2, 0, 3, 0, \dots\}$ are also sequences that diverge but do not diverge to infinity or negative infinity.

The convergence or divergence of a sequence is not affected by the values of any number of its initial terms (whether we omit or change the first 10, the first 1000, or even the first million terms does not matter). From Figure 9.2, we can see that only the part of the sequence that remains after discarding some initial number of terms determines whether the sequence has a limit and the value of that limit when it does exist.

Calculating Limits of Sequences

Since sequences are functions with domain restricted to the positive integers, it is not surprising that the theorems on limits of functions given in Chapter 2 have versions for sequences.

THEOREM 1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$.

1. *Sum Rule:* $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n \rightarrow \infty} (k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

The proof is similar to that of Theorem 1 in Section 2.2 and is omitted.

EXAMPLE 3 By combining Theorem 1 with the limits of Example 1, we have:

- (a) $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = -1 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = -1 \cdot 0 = 0$ Constant Multiple Rule
and Example 1a
- (b) $\lim_{n \rightarrow \infty} \left(\frac{n-1}{n}\right) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{n} = 1 - 0 = 1$ Difference Rule
and Example 1a
- (c) $\lim_{n \rightarrow \infty} \frac{5}{n^2} = 5 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$ Product Rule
- (d) $\lim_{n \rightarrow \infty} \frac{4 - 7n^6}{n^6 + 3} = \lim_{n \rightarrow \infty} \frac{(4/n^6) - 7}{1 + (3/n^6)} = \frac{0 - 7}{1 + 0} = -7.$ Divide numerator and denominator
by n^6 and use the Sum and
Quotient Rules.

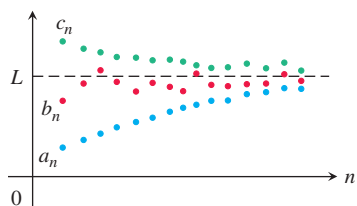


FIGURE 9.4 The terms of sequence $\{b_n\}$ are sandwiched between those of $\{a_n\}$ and $\{c_n\}$, forcing them to the same common limit L .

Be cautious in applying Theorem 1. It does not say, for example, that both of the sequences $\{a_n\}$ and $\{b_n\}$ have limits if their sum $\{a_n + b_n\}$ has a limit. For instance, $\{a_n\} = \{1, 2, 3, \dots\}$ and $\{b_n\} = \{-1, -2, -3, \dots\}$ both diverge, but their sum $\{a_n + b_n\} = \{0, 0, 0, \dots\}$ clearly converges to 0.

One consequence of Theorem 1 is that every nonzero multiple of a divergent sequence $\{a_n\}$ diverges. Suppose, to the contrary, that $\{ca_n\}$ converges for some number $c \neq 0$. Then, by taking $k = 1/c$ in the Constant Multiple Rule in Theorem 1, we see that the sequence

$$\left\{\frac{1}{c} \cdot ca_n\right\} = \{a_n\}$$

converges. Thus, $\{ca_n\}$ cannot converge unless $\{a_n\}$ also converges. If $\{a_n\}$ does not converge, then $\{ca_n\}$ does not converge.

The next theorem is the sequence version of the Sandwich Theorem in Section 2.2. You are asked to prove the theorem in Exercise 119. (See Figure 9.4.)

THEOREM 2—The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N , and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$ also.

An immediate consequence of Theorem 2 is that if $|b_n| \leq c_n$ and $c_n \rightarrow 0$, then $b_n \rightarrow 0$ because $-c_n \leq b_n \leq c_n$. We use this fact in the next example.

EXAMPLE 4 Since $1/n \rightarrow 0$, we know that

- (a) $\frac{\cos n}{n} \rightarrow 0$ because $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$;
 (b) $\frac{1}{2^n} \rightarrow 0$ because $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$;
 (c) $(-1)^n \frac{1}{n} \rightarrow 0$ because $-\frac{1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$.
 (d) If $|a_n| \rightarrow 0$, then $a_n \rightarrow 0$ because $-|a_n| \leq a_n \leq |a_n|$. ■

The application of Theorems 1 and 2 is broadened by a theorem stating that applying a continuous function to a convergent sequence produces a convergent sequence. We state the theorem, leaving the proof as an exercise (Exercise 120).

THEOREM 3—The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \rightarrow L$ and if f is a function that is continuous at L and defined at all a_n , then $f(a_n) \rightarrow f(L)$.

EXAMPLE 5 Show that $\sqrt{(n+1)/n} \rightarrow 1$.

Solution We know that $(n+1)/n \rightarrow 1$. Taking $f(x) = \sqrt{x}$ and $L = 1$ in Theorem 3 gives $\sqrt{(n+1)/n} \rightarrow \sqrt{1} = 1$. ■

EXAMPLE 6 The sequence $\{1/n\}$ converges to 0. By taking $a_n = 1/n$, $f(x) = 2^x$, and $L = 0$ in Theorem 3, we see that $2^{1/n} = f(1/n) \rightarrow f(L) = 2^0 = 1$. The sequence $\{2^{1/n}\}$ converges to 1 (Figure 9.5). ■

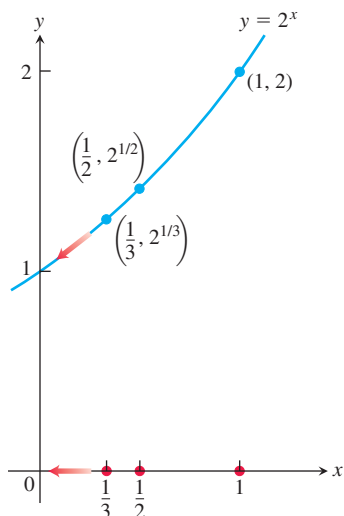


FIGURE 9.5 As $n \rightarrow \infty$, $1/n \rightarrow 0$ and $2^{1/n} \rightarrow 2^0$ (Example 6). The terms of $\{1/n\}$ are shown on the x -axis; the terms of $\{2^{1/n}\}$ are shown as the y -values on the graph of $f(x) = 2^x$.

Using L'Hôpital's Rule

The next theorem formalizes the connection between $\lim_{n \rightarrow \infty} a_n$ and $\lim_{x \rightarrow \infty} f(x)$. It enables us to use l'Hôpital's Rule to find the limits of some sequences.

THEOREM 4 Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{whenever} \quad \lim_{x \rightarrow \infty} f(x) = L.$$

Proof Suppose that $\lim_{x \rightarrow \infty} f(x) = L$. Then for each positive number ε , there is a number M such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M.$$

Let N be an integer that is both greater than M and greater than or equal to n_0 . Since $a_n = f(n)$, it follows that for all $n > N$, we have

$$|a_n - L| = |f(n) - L| < \varepsilon. \quad \blacksquare$$

EXAMPLE 7 Show that

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0.$$

Solution The function $(\ln x)/x$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers. Therefore, by Theorem 4, $\lim_{n \rightarrow \infty} (\ln n)/n$ will equal $\lim_{x \rightarrow \infty} (\ln x)/x$ if the latter exists. A single application of l'Hôpital's Rule shows that

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

We conclude that $\lim_{n \rightarrow \infty} (\ln n)/n = 0$. ■

EXAMPLE 8 Does the sequence whose n th term is

$$a_n = \left(\frac{n+1}{n-1} \right)^n, \quad n \geq 2$$

converge? If so, find $\lim_{n \rightarrow \infty} a_n$.

Solution The function $f(x) = \left(\frac{x+1}{x-1} \right)^x$ is defined for all real numbers $x \geq 2$ and agrees with a_n at all integers $n \geq 2$. If we can show that $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x-1} \right)^x = L$, then by Theorem 4, $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n-1} \right)^n = L$.

The limit $\lim_{x \rightarrow \infty} f(x)$ leads to the indeterminate form 1^∞ . To evaluate this limit, we apply l'Hôpital's Rule after taking the natural logarithm of $f(x)$:

$$\ln f(x) = \ln \left(\frac{x+1}{x-1} \right)^x = x \ln \left(\frac{x+1}{x-1} \right).$$

Then

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln f(x) &= \lim_{x \rightarrow \infty} x \ln \left(\frac{x+1}{x-1} \right) && \infty \cdot 0 \text{ form} \\ &= \lim_{x \rightarrow \infty} \frac{\ln \left(\frac{x+1}{x-1} \right)}{1/x} && \frac{0}{0} \text{ form} \\ &= \lim_{x \rightarrow \infty} \frac{-2/(x^2 - 1)}{-1/x^2} && \text{Apply l'Hôpital's Rule.} \\ &= \lim_{x \rightarrow \infty} \frac{2x^2}{x^2 - 1} = 2. && \text{Simplify and evaluate.} \end{aligned}$$

Therefore, $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x-1} \right)^x = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^2$. Applying Theorem 4, we conclude that the sequence $\{a_n\}$ also converges to e^2 . ■

Commonly Occurring Limits

The next theorem gives some limits that arise frequently.

Factorial Notation

The notation $n!$ (“ n factorial”) means the product $1 \cdot 2 \cdot 3 \cdots n$ of the integers from 1 to n . Notice that $(n + 1)! = (n + 1) \cdot n!$. Thus, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ and $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5 \cdot 4! = 120$. We define $0!$ to be 1. Factorials grow even faster than exponentials, as the table suggests. The values in the table are rounded.

n	e^n	$n!$
1	3	1
5	148	120
10	22,026	3,628,800
20	4.9×10^8	2.4×10^{18}

THEOREM 5 The following six sequences converge to the limits listed below.

$$1. \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

$$2. \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

$$3. \lim_{n \rightarrow \infty} x^{1/n} = 1 \quad (x > 0)$$

$$4. \lim_{n \rightarrow \infty} x^n = 0 \quad (|x| < 1)$$

$$5. \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$$

$$6. \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$$

In Formulas (3) through (6), x remains fixed as $n \rightarrow \infty$.

Proof The first limit was computed in Example 7. The next two can be proved by taking logarithms and applying Theorem 4 (Exercises 117 and 118). The remaining proofs are given in Appendix A.6. ■

EXAMPLE 9 These are examples involving the limits in Theorem 5.

$$(a) \frac{\ln(n^2)}{n} = \frac{2 \ln n}{n} \rightarrow 2 \cdot 0 = 0$$

Formula 1

$$(b) \sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \rightarrow (1)^2 = 1$$

Formula 2

$$(c) \sqrt[3]{3n} = 3^{1/n} (n^{1/n}) \rightarrow 1 \cdot 1 = 1$$

Formula 3 with $x = 3$ and Formula 2

$$(d) \left(-\frac{1}{2}\right)^n \rightarrow 0$$

Formula 4 with $x = -\frac{1}{2}$

$$(e) \left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \rightarrow e^{-2}$$

Formula 5 with $x = -2$

$$(f) \frac{100^n}{n!} \rightarrow 0$$

Formula 6 with $x = 100$ ■

Recursive Definitions

So far, we have calculated each a_n directly from the value of n . But sequences are often defined **recursively** by giving

1. The value(s) of the initial term or terms, and
2. a rule called a **recursion formula** for calculating any later term from terms that precede it.

EXAMPLE 10

- (a) The statements $a_1 = 1$ and $a_n = a_{n-1} + 1$ for $n > 1$ define the sequence $1, 2, 3, \dots, n, \dots$ of positive integers. With $a_1 = 1$, we have $a_2 = a_1 + 1 = 2$, $a_3 = a_2 + 1 = 3$, and so on.
- (b) The statements $a_1 = 1$ and $a_n = n \cdot a_{n-1}$ for $n > 1$ define the sequence $1, 2, 6, 24, \dots, n!, \dots$ of factorials. With $a_1 = 1$, we have $a_2 = 2 \cdot a_1 = 2$, $a_3 = 3 \cdot a_2 = 6$, $a_4 = 4 \cdot a_3 = 24$, and so on.
- (c) The statements $a_1 = 1, a_2 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for $n > 2$ define the sequence $1, 1, 2, 3, 5, \dots$ of **Fibonacci numbers**. With $a_1 = 1$ and $a_2 = 1$, we have $a_3 = 1 + 1 = 2$, $a_4 = 2 + 1 = 3$, $a_5 = 3 + 2 = 5$, and so on.

- (d) As we can see by applying Newton's method (see Exercise 145), the statements $x_0 = 1$ and $x_{n+1} = x_n - [(\sin x_n - x_n^2)/(\cos x_n - 2x_n)]$ for $n > 0$ define a sequence that, when it converges, gives a solution to the equation $\sin x - x^2 = 0$. ■

Bounded Monotonic Sequences

Two concepts that play a key role in determining the convergence of a sequence are those of a *bounded* sequence and a *monotonic* sequence. First we define bounded sequences.

DEFINITION A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n . The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n . The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

EXAMPLE 11

- (a) The sequence $1, 2, 3, \dots, n, \dots$ has no upper bound because it eventually surpasses every number M . However, it is bounded below by every real number less than or equal to 1. The number $m = 1$ is the greatest lower bound of the sequence.
- (b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$ is bounded above by every real number greater than or equal to 1. The upper bound $M = 1$ is the least upper bound (Exercise 137). The sequence is also bounded below by every number less than or equal to $\frac{1}{2}$, which is its greatest lower bound. ■

Convergent sequences are bounded.

Suppose that a sequence $\{a_n\}$ converges to a number L . Applying the definition of convergence with the particular value $\varepsilon = 1$, there must exist a number N such that $|a_n - L| < 1$ if $n > N$. That is,

$$L - 1 < a_n < L + 1 \quad \text{for } n > N.$$

If M is a number larger than both $L + 1$ and all of the finitely many numbers $a_1, a_2, \dots, a_N$, then for every index n we have $a_n \leq M$, and therefore $\{a_n\}$ is bounded from above. Similarly, if m is a number smaller than both $L - 1$ and all of the numbers $a_1, a_2, \dots, a_N$, then m is a lower bound for the sequence. Therefore, all convergent sequences are bounded.

Although it is true that every convergent sequence is bounded, there are bounded sequences that fail to converge. One example is the bounded sequence $\{(-1)^{n+1}\}$ discussed in Example 2. The problem here is that some bounded sequences bounce around in the band determined by any lower bound m and any upper bound M but do not converge (Figure 9.6). An important type of sequence that does not behave that way is one for which each term is at least as large, or at least as small, as its predecessor.

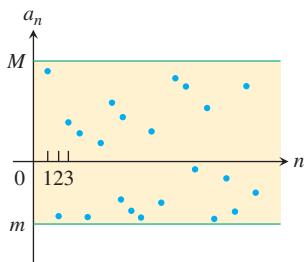


FIGURE 9.6 Some bounded sequences bounce around between their bounds and fail to converge to any limiting value.

DEFINITIONS A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1}$ for all n . That is, $a_1 \leq a_2 \leq a_3 \leq \dots$. The sequence is **nonincreasing** if $a_n \geq a_{n+1}$ for all n . The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

EXAMPLE 12

- (a) The sequence $1, 2, 3, \dots, n, \dots$ is nondecreasing.
- (b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$ is nondecreasing.
- (c) The sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots$ is nonincreasing.
- (d) The constant sequence $3, 3, 3, \dots, 3, \dots$ is both nondecreasing and nonincreasing.
- (e) The sequence $1, -1, 1, -1, 1, -1, \dots$ is not monotonic. ■

A sequence that is bounded from above always has a least upper bound. Likewise, a sequence bounded from below always has a greatest lower bound. These results are based on the *completeness property* of the real numbers, discussed in Appendix A.9. We now prove that if L is the least upper bound of a nondecreasing sequence, then the sequence converges to L , and that if L is the greatest lower bound of a nonincreasing sequence, then the sequence converges to L .

THEOREM 6—The Monotonic Sequence Theorem

If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

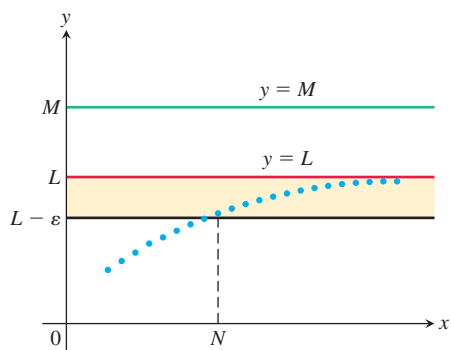


FIGURE 9.7 If the terms of a nondecreasing sequence have an upper bound M , then they have a limit $L \leq M$.

Proof Suppose $\{a_n\}$ is nondecreasing, L is its least upper bound, and we plot the points $(1, a_1), (2, a_2), \dots, (n, a_n), \dots$ in the xy -plane. If M is an upper bound of the sequence, all these points will lie on or below the line $y = M$ (Figure 9.7). The line $y = L$ is the lowest such line. None of the points (n, a_n) lies above $y = L$, but some do lie above any lower line $y = L - \varepsilon$, if ε is a positive number (because $L - \varepsilon$ is not an upper bound). That is,

- a. $a_n \leq L$ for all values of n , and
- b. given any $\varepsilon > 0$, there exists at least one integer N for which $a_N > L - \varepsilon$.

The fact that $\{a_n\}$ is nondecreasing tells us further that

$$a_n \geq a_N > L - \varepsilon \quad \text{for all } n \geq N.$$

Thus, all the numbers a_n beyond the N th number lie within ε of L . This is precisely the condition for L to be the limit of the sequence $\{a_n\}$.

The proof for nonincreasing sequences bounded from below is similar. ■

It is important to realize that Theorem 6 does not say that convergent sequences are monotonic. The sequence $\{(-1)^{n+1}/n\}$ converges and is bounded, but it is not monotonic since it alternates between positive and negative values as it tends toward zero. What the theorem does say is that a nondecreasing sequence converges when it is bounded from above, but it diverges to infinity otherwise.

EXERCISES 9.1**Finding Terms of a Sequence**

Each of Exercises 1–6 gives a formula for the n th term a_n of a sequence $\{a_n\}$. Find the values of a_1, a_2, a_3 , and a_4 .

1. $a_n = \frac{1-n}{n^2}$

2. $a_n = \frac{1}{n!}$

3. $a_n = \frac{(-1)^{n+1}}{2n-1}$

4. $a_n = 2 + (-1)^n$

5. $a_n = \frac{2^n}{2^{n+1}}$

6. $a_n = \frac{2^n - 1}{2^n}$

Each of Exercises 7–12 gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

7. $a_1 = 1, \quad a_{n+1} = a_n + (1/2^n)$

8. $a_1 = 1, \quad a_{n+1} = a_n/(n+1)$

88. $a_n = n\left(1 - \cos \frac{1}{n}\right)$ 89. $a_n = \sqrt{n} \sin \frac{1}{\sqrt{n}}$
 90. $a_n = (3^n + 5^n)^{1/n}$ 91. $a_n = \tan^{-1} n$
 92. $a_n = \frac{1}{\sqrt{n}} \arctan n$ 93. $a_n = \left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2^n}}$
 94. $a_n = \sqrt[n]{n^2 + n}$ 95. $a_n = \frac{(\ln n)^{200}}{n}$
 96. $a_n = \frac{(\ln n)^5}{\sqrt{n}}$ 97. $a_n = n - \sqrt{n^2 - n}$
 98. $a_n = \frac{1}{\sqrt{n^2 - 1} - \sqrt{n^2 + n}}$
 99. $a_n = \frac{1}{n} \int_1^n \frac{1}{x} dx$ 100. $a_n = \int_1^n \frac{1}{x^p} dx, \quad p > 1$

Recursively Defined Sequences

In Exercises 101–108, assume that each sequence converges and find its limit.

101. $a_1 = 2, \quad a_{n+1} = \frac{72}{1 + a_n}$
 102. $a_1 = -1, \quad a_{n+1} = \frac{a_n + 6}{a_n + 2}$
 103. $a_1 = -4, \quad a_{n+1} = \sqrt{8 + 2a_n}$
 104. $a_1 = 0, \quad a_{n+1} = \sqrt{8 + 2a_n}$
 105. $a_1 = 5, \quad a_{n+1} = \sqrt{5a_n}$
 106. $a_1 = 3, \quad a_{n+1} = 12 - \sqrt{a_n}$
 107. $2, 2 + \frac{1}{2}, 2 + \frac{1}{2 + \frac{1}{2}}, 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \dots$
 108. $\sqrt{1}, \sqrt{1 + \sqrt{1}}, \sqrt{1 + \sqrt{1 + \sqrt{1}}}, \dots$
 $\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}, \dots$

Theory and Examples

109. The first term of a sequence is $x_1 = 1$. Each succeeding term is the sum of all those that come before it:

$$x_{n+1} = x_1 + x_2 + \cdots + x_n.$$

Write out enough early terms of the sequence to deduce a general formula for x_n that holds for $n \geq 2$.

110. A sequence of rational numbers is described as follows:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \dots, \frac{a}{b}, \frac{a+2b}{a+b}, \dots$$

Here the numerators form one sequence, the denominators form a second sequence, and their ratios form a third sequence. Let x_n and y_n be, respectively, the numerator and the denominator of the n th fraction $r_n = x_n/y_n$.

- a. Verify that $x_1^2 - 2y_1^2 = -1$, that $x_2^2 - 2y_2^2 = +1$, and, more generally, that if $a^2 - 2b^2 = -1$ or $+1$, then

$$(a + 2b)^2 - 2(a + b)^2 = +1 \quad \text{or} \quad -1,$$

respectively.

- b. The fractions $r_n = x_n/y_n$ approach a limit as n increases. What is that limit? (*Hint:* Use part (a) to show that $r_n^2 - 2 = \pm(1/y_n)^2$ and that y_n is not less than n .)

111. **Newton's method** The following sequences come from the recursion formula for Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Do the sequences converge? If so, to what value? In each case, begin by identifying the function f that generates the sequence.

- a. $x_0 = 1, \quad x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$
 b. $x_0 = 1, \quad x_{n+1} = x_n - \frac{\tan x_n - 1}{\sec^2 x_n}$
 c. $x_0 = 1, \quad x_{n+1} = x_n - 1$

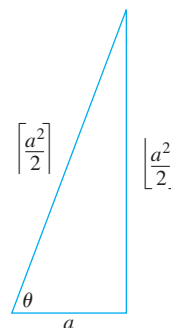
112. a. Suppose that $f(x)$ is differentiable for all x in $[0, 1]$ and that $f(0) = 0$. Define sequence $\{a_n\}$ by the rule $a_n = n f(1/n)$. Show that $\lim_{n \rightarrow \infty} a_n = f'(0)$. Use the result in part (a) to find the limits of the following sequences $\{a_n\}$.

- b. $a_n = n \tan^{-1} \frac{1}{n}$ c. $a_n = n(e^{1/n} - 1)$
 d. $a_n = n \ln\left(1 + \frac{2}{n}\right)$

113. **Pythagorean triples** A triple of positive integers a , b , and c is called a **Pythagorean triple** if $a^2 + b^2 = c^2$. Let a be an odd positive integer and let

$$b = \left\lfloor \frac{a^2}{2} \right\rfloor \quad \text{and} \quad c = \left\lceil \frac{a^2}{2} \right\rceil$$

be, respectively, the integer floor and ceiling for $a^2/2$.



- a. Show that $a^2 + b^2 = c^2$. (*Hint:* Let $a = 2n + 1$ and express b and c in terms of n .)
 b. By direct calculation, or by appealing to the accompanying figure, find

$$\lim_{a \rightarrow \infty} \frac{\left\lfloor \frac{a^2}{2} \right\rfloor}{\left\lceil \frac{a^2}{2} \right\rceil}.$$

114. The n th root of $n!$

- a. Show that $\lim_{n \rightarrow \infty} (2n\pi)^{1/(2n)} = 1$ and hence, using Stirling's approximation (Chapter 8, Additional Exercise 44a), that

$$\sqrt[n]{n!} \approx \frac{n}{e} \quad \text{for large values of } n.$$

- T** b. Test the approximation in part (a) for $n = 40, 50, 60, \dots$, as far as your calculator will allow.

- 115. a.** Assuming that $\lim_{n \rightarrow \infty} (1/n^c) = 0$ if c is any positive constant, show that

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n^c} = 0$$

if c is any positive constant.

- b. Prove that $\lim_{n \rightarrow \infty} (1/n^c) = 0$ if c is any positive constant.

(Hint: If $\varepsilon = 0.001$ and $c = 0.04$, how large should N be to ensure that $|1/n^c - 0| < \varepsilon$ if $n > N$?)

- 116. The zipper theorem** Prove the “zipper theorem” for sequences: If $\{a_n\}$ and $\{b_n\}$ both converge to L , then the sequence

$$a_1, b_1, a_2, b_2, \dots, a_n, b_n, \dots$$

converges to L .

- 117.** Prove that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

- 118.** Prove that $\lim_{n \rightarrow \infty} x^{1/n} = 1$, ($x > 0$).

- 119.** Prove Theorem 2.

- 120.** Prove Theorem 3.

In Exercises 121–124, determine whether the sequence is monotonic and whether it is bounded.

121. $a_n = \frac{3n+1}{n+1}$

122. $a_n = \frac{(2n+3)!}{(n+1)!}$

123. $a_n = \frac{2^n 3^n}{n!}$

124. $a_n = 2 - \frac{2}{n} - \frac{1}{2^n}$

In Exercises 125–134, determine whether the sequence is monotonic, whether it is bounded, and whether it converges.

125. $a_n = 1 - \frac{1}{n}$

126. $a_n = n - \frac{1}{n}$

127. $a_n = \frac{2^n - 1}{2^n}$

128. $a_n = \frac{2^n - 1}{3^n}$

129. $a_n = ((-1)^n + 1) \left(\frac{n+1}{n} \right)$

- 130.** The first term of a sequence is $x_1 = \cos(1)$. The next terms are $x_2 = x_1$ or $\cos(2)$, whichever is larger; and $x_3 = x_2$ or $\cos(3)$, whichever is larger (farther to the right). In general,

$$x_{n+1} = \max\{x_n, \cos(n+1)\}.$$

131. $a_n = \frac{1 + \sqrt{2n}}{\sqrt{n}}$

132. $a_n = \frac{n+1}{n}$

133. $a_n = \frac{4^{n+1} + 3^n}{4^n}$

134. $a_1 = 1, \quad a_{n+1} = 2a_n - 3$

In Exercises 135–136, use the definition of convergence to prove the given limit.

135. $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$

136. $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n^2} \right) = 1$

- 137. The sequence $\{n/(n+1)\}$ has a least upper bound of 1** Show that if M is a number less than 1, then the terms of $\{n/(n+1)\}$ eventually exceed M . That is, if $M < 1$, there is an integer N such that $n/(n+1) > M$ whenever $n > N$. Since $n/(n+1) < 1$ for every n , this proves that 1 is a least upper bound for $\{n/(n+1)\}$.

- 138. Uniqueness of least upper bounds** Show that if M_1 and M_2 are least upper bounds for the sequence $\{a_n\}$, then $M_1 = M_2$. That is, a sequence cannot have two different least upper bounds.

- 139.** Is it true that a sequence $\{a_n\}$ of positive numbers must converge if it is bounded from above? Give reasons for your answer.

- 140.** Prove that if $\{a_n\}$ is a convergent sequence, then to every positive number ε there corresponds an integer N such that

$$|a_m - a_n| < \varepsilon \quad \text{whenever } m > N \quad \text{and } n > N.$$

- 141. Uniqueness of limits** Prove that limits of sequences are unique. That is, show that if L_1 and L_2 are numbers such that $a_n \rightarrow L_1$ and $a_n \rightarrow L_2$, then $L_1 = L_2$.

- 142. Limits and subsequences** If the terms of one sequence appear in another sequence in their given order, we call the first sequence a **subsequence** of the second. Prove that if two subsequences of a sequence $\{a_n\}$ have different limits $L_1 \neq L_2$, then $\{a_n\}$ diverges.

- 143.** For a sequence $\{a_n\}$ the terms of even index are denoted by a_{2k} and the terms of odd index by a_{2k+1} . Prove that if $a_{2k} \rightarrow L$ and $a_{2k+1} \rightarrow L$, then $a_n \rightarrow L$.

- 144.** Prove that a sequence $\{a_n\}$ converges to 0 if and only if the sequence of absolute values $\{|a_n|\}$ converges to 0.

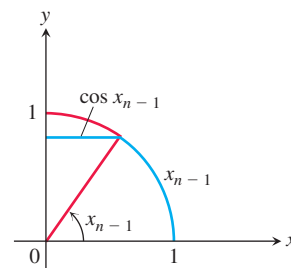
- 145. Sequences generated by Newton's method** Newton's method, applied to a differentiable function $f(x)$, begins with a starting value x_0 and constructs from it a sequence of numbers $\{x_n\}$ that under favorable circumstances converges to a zero of f . The recursion formula for the sequence is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

- a. Show that the recursion formula for $f(x) = x^2 - a$, $a > 0$, can be written as $x_{n+1} = (x_n + a/x_n)/2$.

- T** b. Starting with $x_0 = 1$ and $a = 3$, calculate successive terms of the sequence until the display begins to repeat. What number is being approximated? Explain.

- T** **146. A recursive definition of $\pi/2$** If you start with $x_1 = 1$ and if you define the subsequent terms of $\{x_n\}$ by the rule $x_n = x_{n-1} + \cos x_{n-1}$, you generate a sequence that converges rapidly to $\pi/2$. (a) Try it. (b) Use the accompanying figure to explain why the convergence is so rapid.



COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the sequences in Exercises 147–158.

- a. Calculate and then plot the first 25 terms of the sequence. Does the sequence appear to be bounded from above or below? Does it appear to converge or diverge? If it does converge, what is the limit L ?
- b. If the sequence converges, find an integer N such that $|a_n - L| \leq 0.01$ for $n \geq N$. How far in the sequence do you have to get for the terms to lie within 0.0001 of L ?

147. $a_n = \sqrt[n]{n}$

148. $a_n = \left(1 + \frac{0.5}{n}\right)^n$

149. $a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{5^n}$

150. $a_1 = 1, \quad a_{n+1} = a_n + (-2)^n$

151. $a_n = \sin n$

152. $a_n = n \sin \frac{1}{n}$

153. $a_n = \frac{\sin n}{n}$

154. $a_n = \frac{\ln n}{n}$

155. $a_n = (0.9999)^n$

156. $a_n = (123456)^{1/n}$

157. $a_n = \frac{8^n}{n!}$

158. $a_n = \frac{n^{41}}{19^n}$

9.2 Infinite Series

An *infinite series* is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots.$$

The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it. Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead we look at the result of summing just the first n terms of the sequence. The sum of the first n terms

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is an ordinary finite sum and can be calculated by normal addition. It is called the *n th partial sum*. As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense that the terms of a sequence approach a limit, as discussed in Section 9.1.

For example, to assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots,$$

we add the terms one at a time from the beginning and look for a pattern in how these partial sums grow.

Partial sum		Value	Suggestive expression for partial sum
First:	$s_1 = 1$	1	$2 - 1$
Second:	$s_2 = 1 + \frac{1}{2}$	$\frac{3}{2}$	$2 - \frac{1}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4}$	$2 - \frac{1}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n th:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$

Indeed there is a pattern. The partial sums form a sequence whose n th term is

$$s_n = 2 - \frac{1}{2^{n-1}}.$$